Emily Riehl

Johns Hopkins University

The complicial sets model of higher ∞-categories

Structures supérieures
Centre International de Rencontres Mathématiques

The idea of a higher ∞-category

An ∞-category, a nickname for an $(\infty, 1)$-category, has:

- objects
- 1-arrows between these objects
- with composites of these 1-arrows witnessed by invertible 2-arrows
- with composition associative up to invertible 3-arrows (and unital)
- with these witnesses coherent up to invertible arrows all the way up

A higher ∞-category, meaning an (∞, n)-category for $0 \leq n \leq \infty$, has:

- objects
- 1-arrows between these objects
- 2-arrows between these 1-arrows
- $\vdots$
- n-arrows between these $n-1$-arrows
- plus higher invertible arrows witnessing composition, units, associativity, and coherence all the way up

Fully extended topological quantum field theories

The (∞, n)-category Bord_n has

- objects = compact 0-manifolds
- k-arrows = k-manifolds with corners, for $1 \leq k \leq n$
- $n + 1$-arrows = diffeomorphisms of n-manifolds rel boundary
- $n + m + 1$-arrows = m-fold isotopies of diffeomorphisms, $m \geq 1$

often with extra structure (eg framing, orientation, G-structure).

A fully extended topological quantum field theory is a homomorphism with domain Bord_n, preserving the monoidal structure and all compositions. The cobordism hypothesis classifies fully extended TQFTs of framed bordisms by the value taken by the positively oriented point.

Dan Freed

- The cobordism hypothesis, Bulletin of the AMS, vol 50, no 1, 2013, 57–92; arXiv:1210.5100

On the unicity of the theory of higher ∞-categories

The schematic idea of an (∞, n)-category is made rigorous by various models: θ_n-spaces, iterated complete Segal spaces, Segal n-categories, n-quasi-categories, n-relative categories, …

Theorem (Barwick–Schommer-Pries, et al). All of the above models of (∞, n)-categories are equivalent.

Clark Barwick and Christopher Schommer-Pries

- On the Unicity of the Homotopy Theory of Higher Categories arXiv:1112.0040

Thus, it's tempting to work "model independently" when envoking higher ∞-categories.

But the theory of higher ∞-categories has not yet been comprehensively developed in any model, so there is "analytic" work still to be done.

Plan

Goal: introduce a user-friendly model of higher ∞-categories

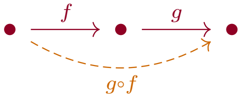
1. A simplicial model of $(\infty, 1)$-categories
2. Towards a simplicial model of $(\infty, 2)$-categories
3. The complicial sets model of higher ∞-categories
4. Complicial sets in the wild (joint with Dominic Verity)

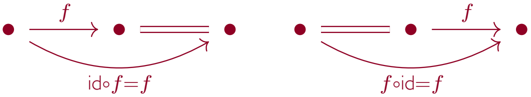
1

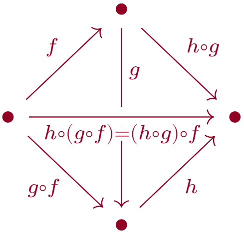
A simplicial model of $(\infty, 1)$-categories

The idea of a 1-category

A 1-category has:

- objects: 
- 1-arrows: 
- composition: 

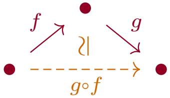
- identity 1-arrows: 
- identity axioms: 

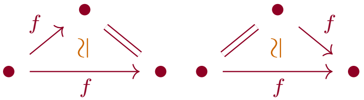
- associativity axioms: 

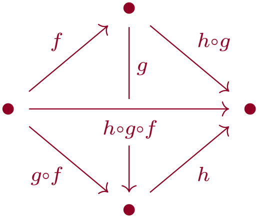
From 1-categories to $(\infty, 1)$-categories

The composition operation and associativity and unit axioms in a 1-category become higher data in an $(\infty, 1)$-category.

An $(\infty, 1)$-category has:

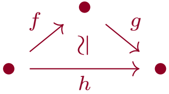
- objects $\bullet$; 1-arrows $\bullet \longrightarrow \bullet$; identity 1-arrows $\bullet = \bullet$
- composition 

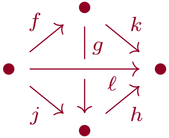
witnessed by invertible 2-arrows
- identity composition witnesses 

- invertible 3-arrows witnessing associativity 

A model for $(\infty, 1)$-categories

In a quasi-category, one popular model for an $(\infty, 1)$-category, this data is structured as a simplicial set with:

- 0-simplices = $\bullet$ = objects
- 1-simplices = $\bullet \longrightarrow \bullet$ = 1-arrows
- 2-simplices = 

= binary composites
- 3-simplices = 

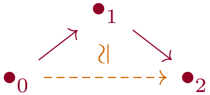
= ternary composites
- n-simplices = n-ary composites
- with degenerate simplices used to encode identity arrows and identity composition witnesses

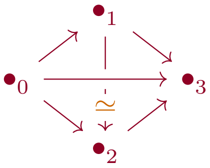
A model for $(\infty, 1)$-categories

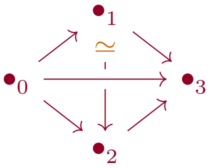
A quasi-category is a "simplicial set with composition": a simplicial set in which every inner horn can be filled to a simplex.

An inner horn is the subcomplex of an n-simplex missing the top cell and the face opposite the vertex $\bullet_k$ for $0 < k < n$.

Low dimensional horn filling:







Exercise: In a quasi-category, all n-arrows with $n > 1$ are equivalences.

Summary: quasi-categories model ∞-categories

A quasi-category is a model of an infinite-dimensional category structured as a simplicial set.

- Basic data is given by low dimensional simplices:
  - 0-simplices = objects
  - 1-simplices = 1-arrows
- Axioms are witnessed by higher simplices:
  - 2-simplices witness binary composites
  - 3-simplices witness associativity of ternary composition
- Higher simplices also regarded as arrows: n-simplices = n-arrows
- Axioms imply that n-arrows are equivalences for $n > 1$.

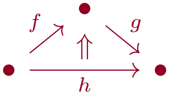
Thus a quasi-category is an $(\infty, 1)$-category, with all n-arrows with $n > 1$ weakly invertible.

2

Towards a simplicial model of $(\infty, 2)$-categories

Towards a simplicial model of an $(\infty, 2)$-category

How might a simplicial set model an $(\infty, 2)$-category?

- 0-simplices = $\bullet$ = objects
- 1-simplices = $\bullet \longrightarrow \bullet$ = 1-arrows
- 2-simplices = 

= 2-arrows

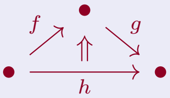
Problem: the 2-simplices must play a dual role, in which they are

- interpreted as inhabited by possibly non-invertible 2-cells
- while also serving as witnesses for composition of 1-simplices

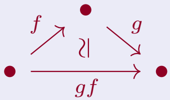
in which case it does not make sense to think of their inhabitants as non-invertible.

Idea: mark the 2-simplex witnesses for composition as "thin" and demand that thin 2-simplices behave like 2-dimensional equivalences.

Towards a simplicial model of an $(\infty, 2)$-category

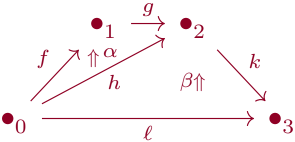
- 2-simplices = 

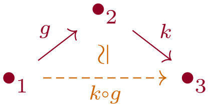
= 2-arrows

- thin 2-simplices 

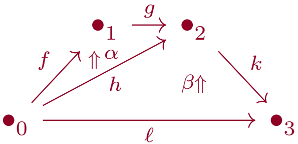
witness 1-arrow composition

Now 3-simplices witness composition of 2-arrows:

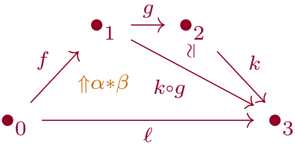
given 

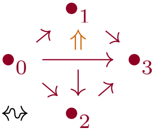
fill 

then fill



$\simeq$





3

The complicial sets model of higher ∞-categories

Marked simplicial sets

For a simplicial set to model a higher ∞-category with non-invertible arrows in each dimension:

- It should have a distinguished set of "thin" n-simplices witnessing composition of $n-1$-simplices.
- Identity arrows, encoded by the degenerate simplices, should be thin.
- Thin simplices should behave like equivalences.
- In particular, 1-simplices that witness an equivalence between objects should also be thin.

This motivates the following definition:

A marked simplicial set is a simplicial set with a designated subset of thin simplices that includes all degenerate simplices.

The symbol "$\simeq$" is used to decorate thin simplices.

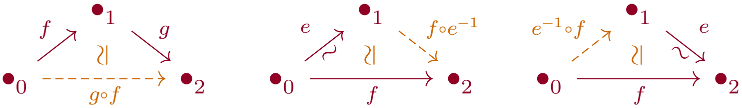
Complicial sets

Recall:

A quasi-category is a “simplicial set with composition”: a simplicial set in which every inner horn can be filled to a simplex.

A complicial set is a “marked simplicial set with composition”: a simplicial set in which every admissible horn can be filled to a simplex and in which thin simplices satisfy the 2-of-3 property.

Low dimensional admissible horn filling:

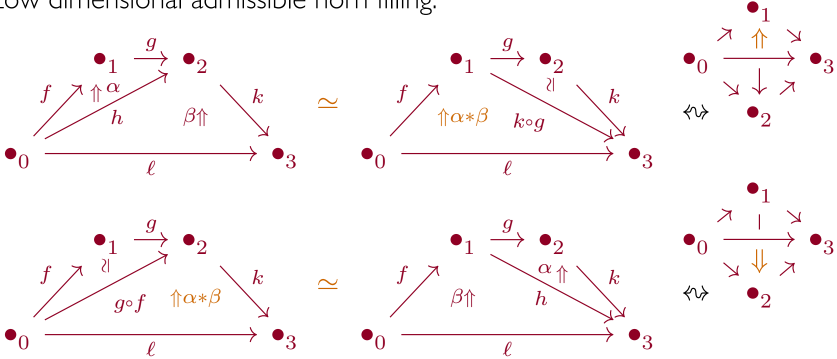


and if two of f, g, and $g \circ f$ are thin so is the third.

Complicial sets

A complicial set is a "marked simplicial set with composition": a simplicial set in which every admissible horn can be filled to a simplex and in which thin simplices satisfy the 2-of-3 property.

Low dimensional admissible horn filling:



and if two of α, β, and $\alpha * \beta$ are thin so is the third.

Admissible horns

An n-simplex in a marked simplicial set is k-admissible — "its kth face is the composite of its $k-1$ and $k+1$-faces" — if every face that contains all of the vertices $\bullet_{k-1}, \bullet_k, \bullet_{k+1}$ is thin.

Thin faces include:

- the n-simplex
- all codimension-1 faces except the $(k-1)$th, kth, and $(k+1)$th
- the 2-simplex spanned by $\{\bullet_{k-1}, \bullet_k, \bullet_{k+1}\}$ when $0 < k < n$
- the edge spanned by $\{\bullet_0, \bullet_1\}$ when $k = 0$ or $\{\bullet_{n-1}, \bullet_n\}$ when $k = n$.

An k-admissible n-horn is the subcomplex of the k-admissible n-simplex that is missing the n-simplex and its k-th face.

Strict ω-categories as strict complicial sets

A strict complicial set is a complicial set in which every admissible horn can be filled uniquely, a "marked simplicial set with unique composition."

Any strict ω-category $\mathcal{C}$ defines a strict complicial set $N\mathcal{C}$, called the Street nerve, whose n-simplices are strict ω-functors

$$\mathcal{O}_n \to \mathcal{C},$$

where

- $\mathcal{O}_n$ is the free strict n-category generated by the n-simplex and
- an n-simplex in $N\mathcal{C}$ is thin just when the ω-functor $\mathcal{O}_n \to \mathcal{C}$ carries the top-dimensional n-arrow in $\mathcal{O}_n$ to an identity in $\mathcal{C}$.

Street-Roberts Conjecture (Verity). The Street nerve defines a fully faithful embedding of strict ω-categories into marked simplicial sets, and the essential image is the category of strict complicial sets.

Strict ω-categories as *weak* complicial sets

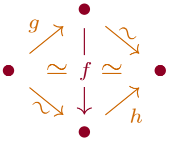
Strict ω-categories can also be a source of *weak* rather than *strict* complicial sets, simply by choosing a more expansive marking convention.

Any strict ω-category $\mathcal{C}$ defines a complicial set $N\mathcal{C}$ whose

- n-simplices are strict ω-functors $\mathcal{O}_n \to \mathcal{C}$ and where
- an n-simplex in $N\mathcal{C}$ is thin just when the ω-functor $\mathcal{O}_n \to \mathcal{C}$ carries the top-dimensional n-arrow in $\mathcal{O}_n$ to an equivalence in $\mathcal{C}$.

Moreover the complicial sets that arise in this way are saturated, meaning that every n-arrow equivalence is thin.

Saturation is a 2-of-6 property for thin simplices:

given an equivalence 

then $f, g, h, h \circ f \circ g$ are also thin.

The n-complicial sets model of (∞, n)-categories

An n-complicial set is a saturated complicial set in which every simplex above dimension n is thin.

For example:

- the nerve of an ordinary 1-groupoid defines a 0-complicial set with everything marked as thin
- the nerve of an ordinary 1-category defines a 1-complicial set with the isomorphisms marked as thin
- the nerve of a strict 2-category defines a 2-complicial set with the 2-arrow isomorphisms and 1-arrow equivalences marked as thin

In fact:

- A 0-complicial set is the same thing as a Kan complex, with everything marked as thin.
- A 1-complicial set is exactly a quasi-category, with the equivalences marked as thin.

Summary: complicial sets model higher ∞-categories

A complicial set is a model of an infinite-dimensional category structured as a marked simplicial set.

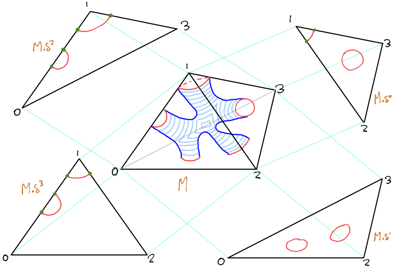
- Basic data is given by simplices:
  - 0-simplices = objects
  - n-simplices = n-arrows
- Axioms are witnessed by thin simplices:
  - thin n-simplices exhibit binary composites of $(n-1)$-simplices
- Thin simplices define invertible arrows:
  - thin n-simplices = n-equivalences
- In a saturated complicial set, all equivalences are thin.

An n-complicial set, a saturated complicial set in which every simplex above dimension n is thin, is a model of an (∞, n)-category.

4

Complicial sets in the wild (joint with Dominic Verity)

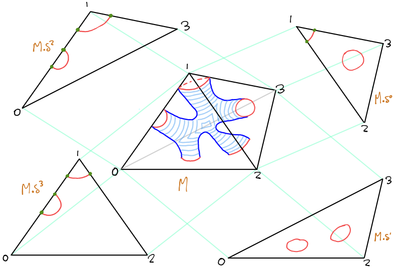
A simplicial set of simplicial bordisms (Verity)



A n-simplicial bordism is a functor from the category of faces of the n-simplex to the category of PL-manifolds and regular embeddings satisfying a boundary condition.

- Simplicial bordisms assemble into a semi simplicial set that admits fillers for all horns, constructed by gluing in cylinders.
- By a theorem of Rourke–Sanderson, degenerate simplices exist and make simplicial bordisms into a genuine Kan complex.

A complicial set of simplicial bordisms (Verity)



The Kan complex of simplicial bordisms can be marked in various ways:

- mark all bordisms as equivalences
- mark only trivial bordisms, which collapse onto their odd faces
- mark the simplicial bordisms that define h-cobordisms from their odd to their even faces

Theorem (Verity). All three marking conventions turn simplicial bordisms into a complicial set, and the third contains the saturation of the second.

Complicial sets defined as homotopy coherent nerves

The homotopy coherent nerve converts a simplicially enriched category into a simplicial set.

Theorem (Cordier–Porter). The homotopy coherent nerve of a Kan complex enriched category is a quasi-category.

Theorem (Cordier–Porter). The homotopy coherent nerve of a 0-complicial set enriched category is a 1-complicial set.

Similarly:

Theorem*(Verity). The homotopy coherent nerve of a n-complicial set enriched category is a $n+1$-complicial set.

In particular, there are a plethora of 2-complicial sets of ∞-categories …

The analytic vs synthetic theory of ∞-categories

The notion of an ∞-category is made rigorous by various models.

Q: How might you develop the category theory of ∞-categories?

Strategies:

- work analytically to give categorical definitions and prove theorems using the combinatorics of one model

  (eg., Joyal, Lurie, Gepner-Haugseng, Cisinski in $q\mathcal{C}at$; Kazhdan-Varshavsky, Rasekh in $\mathcal{R}ezk$; Simpson in $\mathcal{S}egal$)

- work synthetically to give categorical definitions and prove theorems in various models $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ at once

  (R-Verity: an ∞-cosmos axiomatizes the common features of the categories $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ of ∞-categories)

- work synthetically in a simplicial type theory augmenting homotopy type theory to prove theorems in $\mathcal{R}ezk$

  (R-Shulman: an ∞-category is a type with unique binary composites in which isomorphism is equivalent to identity)

∞-cosmoi of ∞-categories

Idea: an ∞-cosmos is a category in which ∞-categories live as objects that has enough structure to develop "formal category theory."

An ∞-cosmos is*:

- a quasi-categorically enriched category
- admitting "strict homotopy limits": flexible weighted simplicially enriched limits.

Examples of ∞-cosmoi:

- models of $(\infty, 1)$-categories: $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, 1-$\mathcal{C}omp$
- models of (∞, n)-categories: n-$q\mathcal{C}at$, θ_n-$\mathcal{S}p$, $\mathcal{CSS}_n$, n-$\mathcal{C}omp$
- $\mathcal{C}at$, $\mathcal{K}an$, $\mathcal{C}omp$
- If $\mathcal{K}$ is an ∞-cosmos, so are $\mathcal{C}art(\mathcal{K})$, $co\mathcal{C}art(\mathcal{K})$ as well as the slices $\mathcal{K}_{/B}$, $\mathcal{C}art(\mathcal{K})_{/B}$, $co\mathcal{C}art(\mathcal{K})_{/B}$ over an ∞-category B.

Why all the fuss about co/cartesian fibrations?

Challenge: define the Yoneda embedding as a functor between ∞-categories.

- Why is this so onerous? It's difficult to fully specify the data of a homotopy coherent diagram.
- Instead, an ∞-category-valued diagram can be repackaged as a co/cartesian fibration, with the homotopy coherence encoded by a universal property.

Idea: a co/cartesian fibration $E \overset{p}{\twoheadrightarrow} B$ is a family of ∞-categories E_b parametrized covariantly/contravariently by elements b of B.

The synthetic definition of a cocartesian fibration: a functor $E \overset{p}{\twoheadrightarrow} B$ so that $E^{2} \overset{p}{\twoheadrightarrow} p{\downarrow}B$ admits a left adjoint right inverse.

The global universal property of co/cartesian fibrations

The codomain projection functor $\text{cod}\colon \text{co}\mathcal{C}\text{art}(\mathcal{K}) \to \mathcal{K}$ defines a "cartesian fibration of quasi-categorically enriched categories":

- For $F \overset{q}{\twoheadrightarrow} A$ and $E \overset{p}{\twoheadrightarrow} B$ in $\text{co}\mathcal{C}\text{art}(\mathcal{K})$, the map $\text{Fun}^{\text{cart}}(q,p) \twoheadrightarrow \text{Fun}(A,B)$ defines a cocartesian fibration in $\text{q}\mathcal{C}\text{at}$.
- Pre- or post-composing by the arrows of $\text{co}\mathcal{C}\text{art}(\mathcal{K})$ defines a cartesian functor between these cocartesian fibrations.
- A pullback $\begin{array}{ccc} F & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$ forms a "cartesian lift of f with codomain p."

Consequently, for $U \hookrightarrow V$, cocartesian cocones with nadir p

$$\begin{array}{ccc} \mathfrak{C}U^{\triangleright} & \longrightarrow & \text{co}\mathcal{C}\text{art}(\mathcal{K}) \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle \text{cod}} \\ \mathfrak{C}V^{\triangleright} & \longrightarrow & \mathcal{K} \end{array}$$

admit extensions that are unique up to a contractible space of choices.

The comprehension construction

A canonical lifting problem defines the comprehension construction:

$$\begin{array}{ccccccc} & & \mathfrak{C}\emptyset^{\triangleright} & \xrightarrow{p\colon E \twoheadrightarrow B} & \mathrm{co}\mathfrak{C}\mathrm{art}(\mathcal{K}) & \overset{\mathrm{dom}}{\twoheadrightarrow} & \mathcal{K} \\ & & \big\downarrow & \nearrow & \big\downarrow \mathrm{cod} & & \\ \mathfrak{C}B_0 & \hookrightarrow & \mathfrak{C}B_0^{\triangleright} & \longrightarrow & \mathcal{K} & & \end{array}$$

which "straightens" p into a homotopy coherent diagram $c_p\colon \mathfrak{C}B_0 \to \mathcal{K}$ indexed by the underlying quasi-category of B.

Applying comprehension in $\mathcal{K}_{/A}$ to a universal fibration $\tilde{U} \overset{\pi}{\twoheadrightarrow} U$ in $\mathcal{K}$, yields $c_\pi\colon \mathfrak{C}\mathrm{Fun}(A,U) \to \mathrm{co}\mathfrak{C}\mathrm{art}(\mathcal{K})_{/A}$, which "unstraightens" an ∞-category-valued diagram into a cocartesian fibration over A.

Applying comprehension in $\mathfrak{C}\mathrm{art}(\mathcal{K})_{/A}$ to the cocartesian fibration $A^2 \overset{\mathrm{cod}}{\twoheadrightarrow} A$ constructs the Yoneda embedding $\mathfrak{C}A_0 \to \mathfrak{C}\mathrm{art}(\mathcal{K})_{/A}$.

References

For more on the complicial sets model of higher ∞-categories see:

Dominic Verity

- Complicial sets, characterising the simplicial nerves of strict ω-categories, Mem. Amer. Math. Soc., 2008; arXiv:math/0410412
- Weak complicial sets I, basic homotopy theory, Adv. Math., 2008; arXiv:math/0604414
- Weak complicial sets II, nerves of complicial Gray-categories, Contemporary Mathematics, 2007, arXiv:math/0604416

Emily Riehl

- Complicial sets, an overture, 2016 MATRIX Annals, arXiv:1610.06801

Emily Riehl and Dominic Verity

- Elements of ∞-Category Theory, draft book in progress www.math.jhu.edu/~eriehl/elements.pdf (particularly Appendix D: the combinatorics of (marked) simplicial sets)