A formal language for ∞-category theory

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(joint with Dominic Verity)

Hopkins @ Home

President's Frontier Award Lecture

Goal: To explain the following theorem concerning the **Model-Independence of ∞-Category Theory** proven with the support of the **Frontier Award***.

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Theorem (Riehl–Verity, 2020):
There is a **formal language** that can be used to encode statements about **∞-categories**, and all statements written in this language are **invariant** under **change of model**.

Q: What are the natural numbers?

0, 1, 2, 3, 4, 5, . . . , 17, 18, 19, ...

Peano's Postulates

"Arithmetices principia, nova methodo exposita," 1889

- There is a natural number "0".
- Every natural number has a "successor," also a natural number.
- 0 is not the successor of any natural number.
- No two natural numbers have the same successor.
- Any set that contains 0, and also contains the successor of every natural number it contains, contains all of the natural numbers.

Dedekind "Was sind und was sollen die Zahlen?" 1888

"In science nothing capable of proof should be accepted without proof."

- $0 \in \mathbb{N}$
- $\forall x \in \mathbb{N}, \ succ(x) \in \mathbb{N}$
- $\forall x \in \mathbb{N}, \ succ(x) \neq 0$
- $\forall x, y \in \mathbb{N}, \ succ(x) = succ(y) \rightarrow x = y$
- $\forall P, \ 0 \in P \wedge (\forall x \in \mathbb{N}, \ x \in P \rightarrow succ(x) \in P) \rightarrow \mathbb{N} \subset P$

Some consequences of Peano's Axioms:

- addition $a+b=b+a$ $\quad a+(b+c)=(a+b)+c$ $\quad a+0=a$
- multiplication $a \times b = b \times a$ $\quad a \times (b+c) = (a \times b) + (a \times c)$
- exponentiation $a^{b+c} = a^b \times a^c$ $\quad (a^b)^c = a^{b \times c} = (a^c)^b$
- number theory

Q: Do the natural numbers exist?

Q: Can the natural numbers be defined in set theory?

Von Neumann's construction

$\mathbb{N}_{VN}$

$$0 := \phi := \{\ \}$$

$$1 := \{0\} := \{\phi\}$$

$$2 := \{0,1\} := \{\phi, \{\phi\}\}$$

$$3 := \{0,1,2\} := \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$$

$$4 := \{0,1,2,3\} := \{\phi, \{\phi\}, \{\phi,\{\phi\}\}, \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}\}$$

Zermelo's construction

$\mathbb{N}_{Z}$

$$0 := \phi := \{\ \}$$

$$1 := \{0\} := \{\phi\}$$

$$2 := \{1\} := \{\{\phi\}\}$$

$$3 := \{2\} := \{\{\{\phi\}\}\}$$

$$4 := \{3\} := \{\{\{\{\phi\}\}\}\}$$

Both sets satisfy the Peano postulates and thus define **models** of the natural numbers.

Our two models of the natural numbers are not equal

$$\mathbb{N}_{VN}, \phi, succ(n) := n \cup \{n\}$$

$$\mathbb{N}_{Z}, \phi, succ(n) := \{n\}$$

but there is a sense in which they are the same:

Dedekind's Categoricity Theorem: All triples given by a set $\mathbb{N}$, an element $0 \in \mathbb{N}$, and a function $succ: \mathbb{N} \to \mathbb{N}$ satisfying Peano's postulates are isomorphic*.

* "iso" + "morphic" = same + shape

Corollary: All number theoretic properties of $\mathbb{N}_Z$ hold for $\mathbb{N}_{VN}$ and vice versa.

	0,	1,	2,	3,	4,	...
$\mathbb{N}_{vN}$	$\{\}$,	$\{0\}$,	$\{0,1\}$,	$\{0,1,2\}$,	$\{0,1,2,3\}$,	...
$\mathbb{N}_Z$	$\{\}$,	$\{0\}$,	$\{1\}$,	$\{2\}$,	$\{3\}$,	...

Benacerraf, "What numbers could not be" 1965

Q: Is 3 an element of 17?

TRUE for $\mathbb{N}_{vN}$ but FALSE for $\mathbb{N}_Z$. But this is also a nonsense statement about the natural numbers.

Q: Can we restrict our formal language to only include meaningful statements?

TAKEAWAYS

Observation: To reason about the natural numbers in the classical foundations of mathematics we need a **model**, eg the sets $\mathbb{N}_{vN}$, $\mathbb{N}_Z$

Problem: The models are not unique.

Observation: To understand the uniqueness of the natural numbers, we need a more sophisticated notion of sameness: not **equality** but **isomorphism**.

Problem: Is there a way to restrict the formal language of mathematics to exclude nonsense statements?

Idea: We need a more precise taxonomy of mathematical objects so we don't ask whether $x = y$ if x is a natural number and y is a triangle.

$\rightsquigarrow$ CATEGORY THEORY

Q: What is an **isomorphism** anyway?

A: It's a concept you can define in any **category**.

Definition: A **category** consists of:

objects X, Y, Z of some specified kind and specified **arrows** $X \xrightarrow{f} Y$ between them, together with identity arrows

$$\begin{tikzcd} X \arrow[loop below, "id_X"] \end{tikzcd}$$

and composites

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & & Z \end{tikzcd}$$

Definition: An **isomorphism** between objects A and B in the same category consists of a pair of arrows $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ so that $g \circ f = id_A$ and $f \circ g = id_B$.

Upshot: there is a notion of isomorphism for mathematical objects of all kinds.

Q: So what is an isomorphism between categories?

Example: In the category of matrices **Mat**

- objects are natural numbers k, m, n, and
- an arrow $n \xrightarrow{A} m$ is an $m \times n$ matrix.

In the category of vector spaces **Vect**

- objects are vector spaces $\mathbb{R}^k, \mathbb{R}^m, \mathbb{R}^n$, and
- an arrow $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a linear transformation.

Mat and **Vect** are both objects in the category **CAT** of categories but they are not isomorphic: instead they are equivalent.

Corollary: All category theoretic properties of **Mat** also hold for **Vect**.

Nonsense Q: Does the category have a countably infinite number of objects?

Q: What is an equivalence anyway?

A: It's a concept you can define in any 2-category.

While the equivalences between 2-categories are defined in a 3-category

and the equivalences between 3-categories are defined in a 4-category

and the equivalences between 4-categories are defined in a 5-category

and the equivalences between 5-categories are defined in a 6-category

and so on ...

This leads to the notion of an ∞-category.

Definition: An **∞-category** consists of:

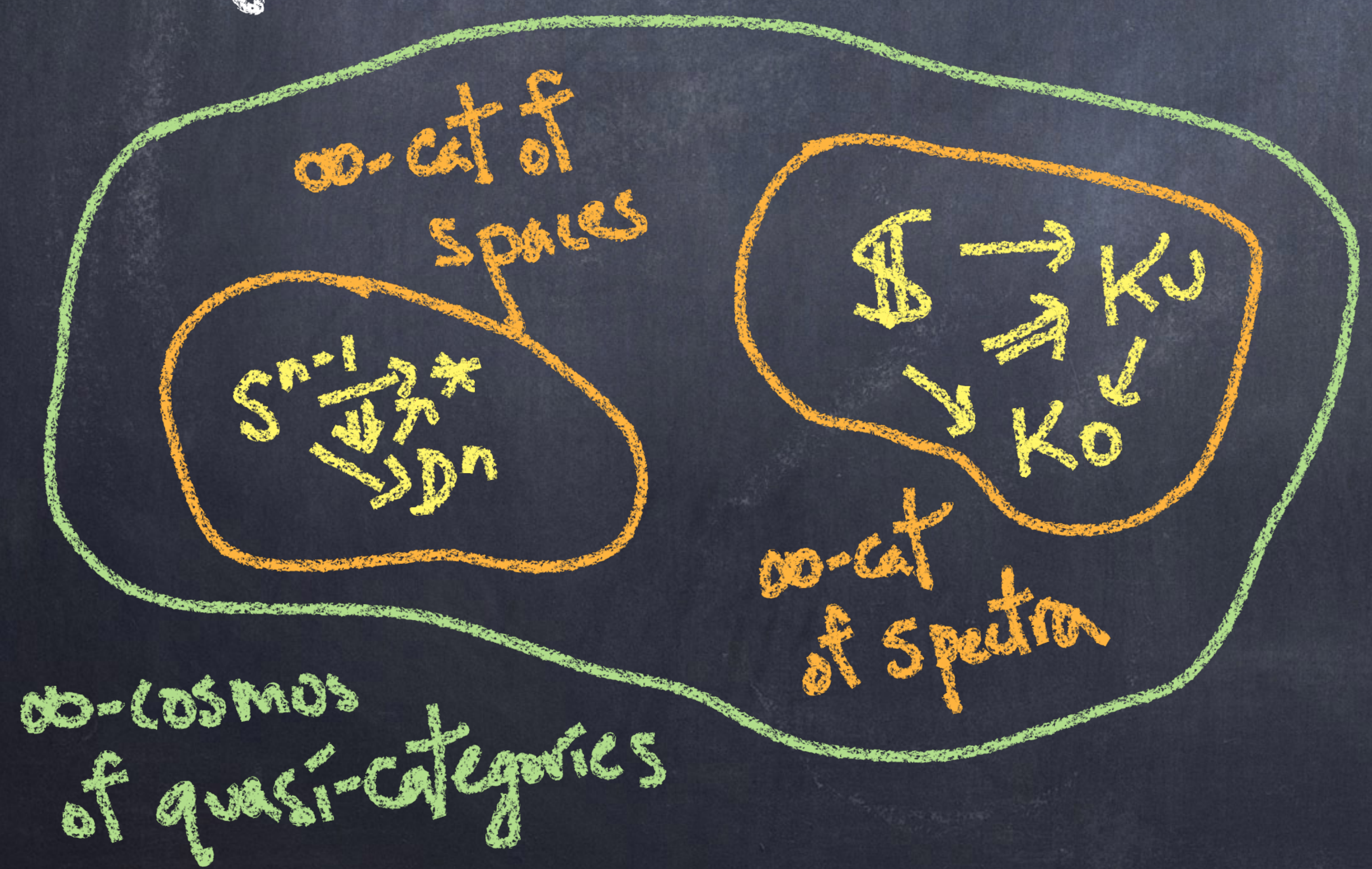
- **objects** X, Y, Z of some specified kind,
- specified **1-arrows** $X \xrightarrow{f} Y$ between them,
- specified **2-arrows** $\begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & Y \end{tikzcd}$ between these,
- and specified **3-arrows** $\begin{tikzcd} X \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] \arrow[Rightarrow, from=U, to=D, shift right=3, "\alpha"'] \arrow[Rightarrow, from=U, to=D, shift left=3, "\beta"] & Y \end{tikzcd}$ $\Gamma: \alpha \Rrightarrow \beta$ between these,

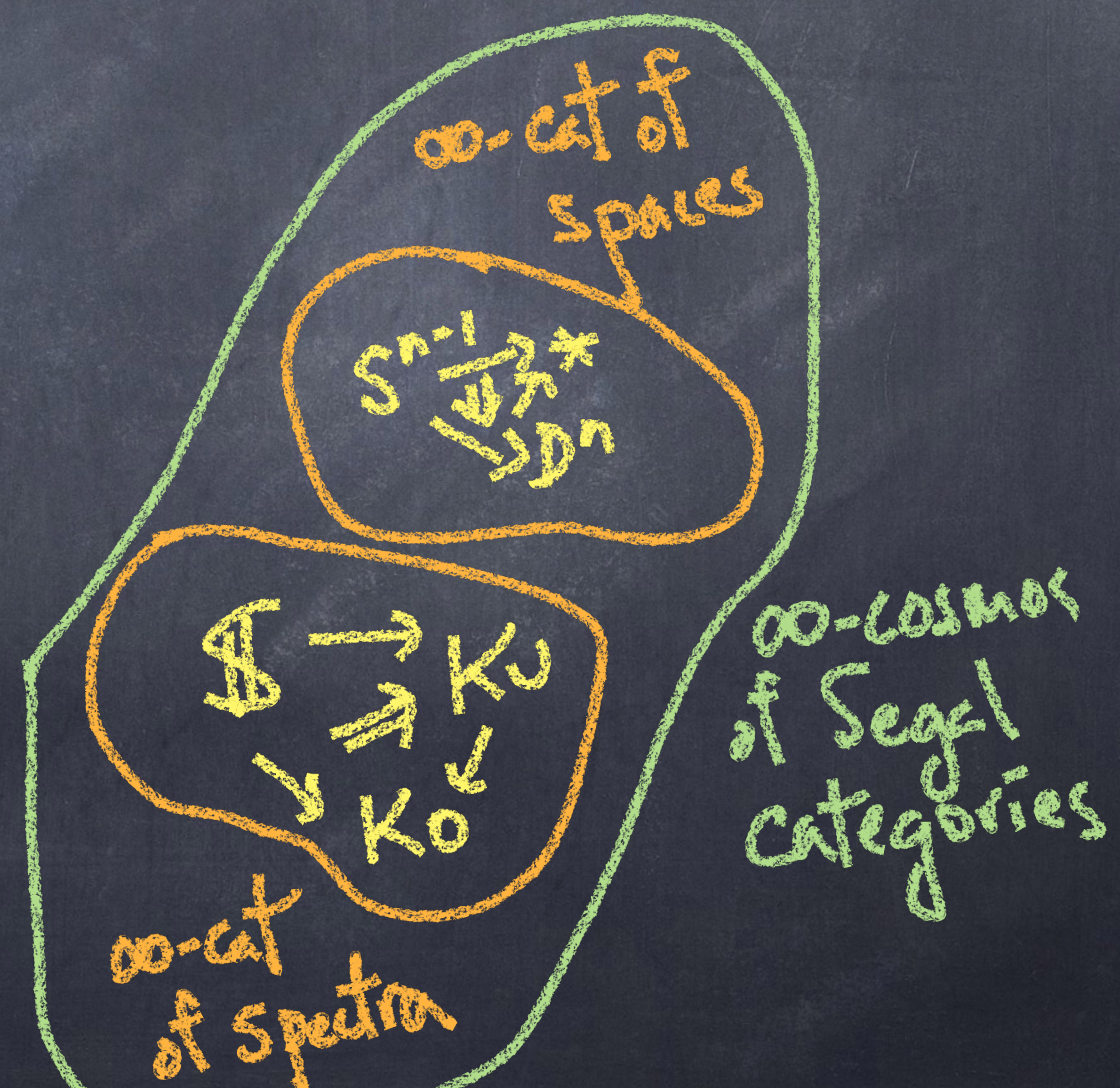
and so on with weak identities and composites in all dimensions

Example: The points and paths and homotopies in any space define an ∞-category.

To prove theorems about ∞-categories in classical foundations we need **models**.

In contrast to models of the natural numbers, models of ∞-categories are really huge, living in a much larger set theoretical universe. We refer to each model as an **∞-cosmos**, since it defines a universe in which ∞-categories live as objects.





Theorem (Riehl–Verity) The theory of ∞-categories can be developed in **any model** and is "**independent**" of the features of that model.

This is a huge help when it comes to proving theorems, as it may be easier to prove a conjecture **analytically** (in the coordinates of a model) as opposed to **synthetically** (from the axioms that define an ∞-cosmos).

But there is still the problem of **evil** statements that are not invariant under equivalence between ∞-categories, much less change of model.

Evil Q: Does an ∞-category have exactly one object?

A formal language for category theory is developed by Makkai in "First Order Logic with Dependent Sorts" 1995, using the following signature to structure the variables.

$$
\begin{tikzcd}
\dot{\mathrm{Comp}} \arrow[dr, shift left=2] \arrow[dr] \arrow[dr, shift right=2] & \dot{\mathrm{I}} \arrow[d] & \dot{\mathrm{E}}\mathrm{q} \arrow[dl, shift left] \arrow[dl, shift right] \\
 & A \arrow[d, shift left] \arrow[d, shift right] & \\
 & O &
\end{tikzcd}
$$

$$\forall x,y\in O,\ \forall f,g\in A(x,y),\ f=g$$

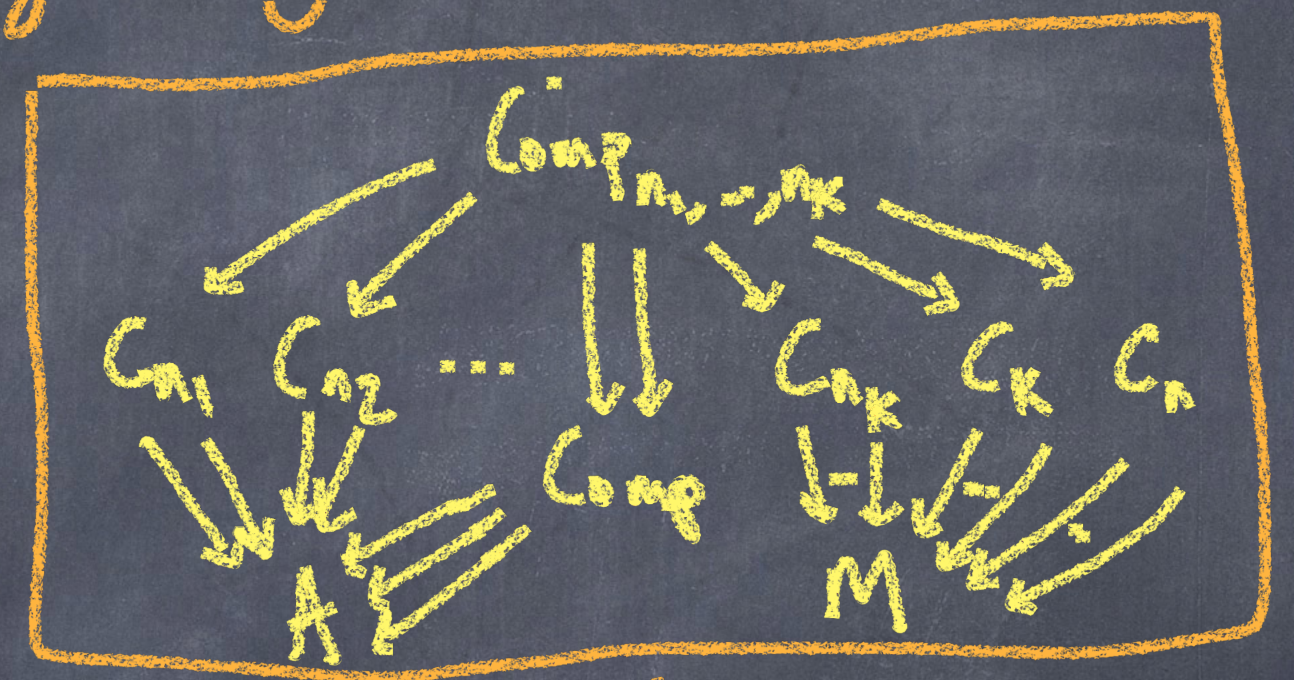
"Between any two objects, there is at most one arrow."

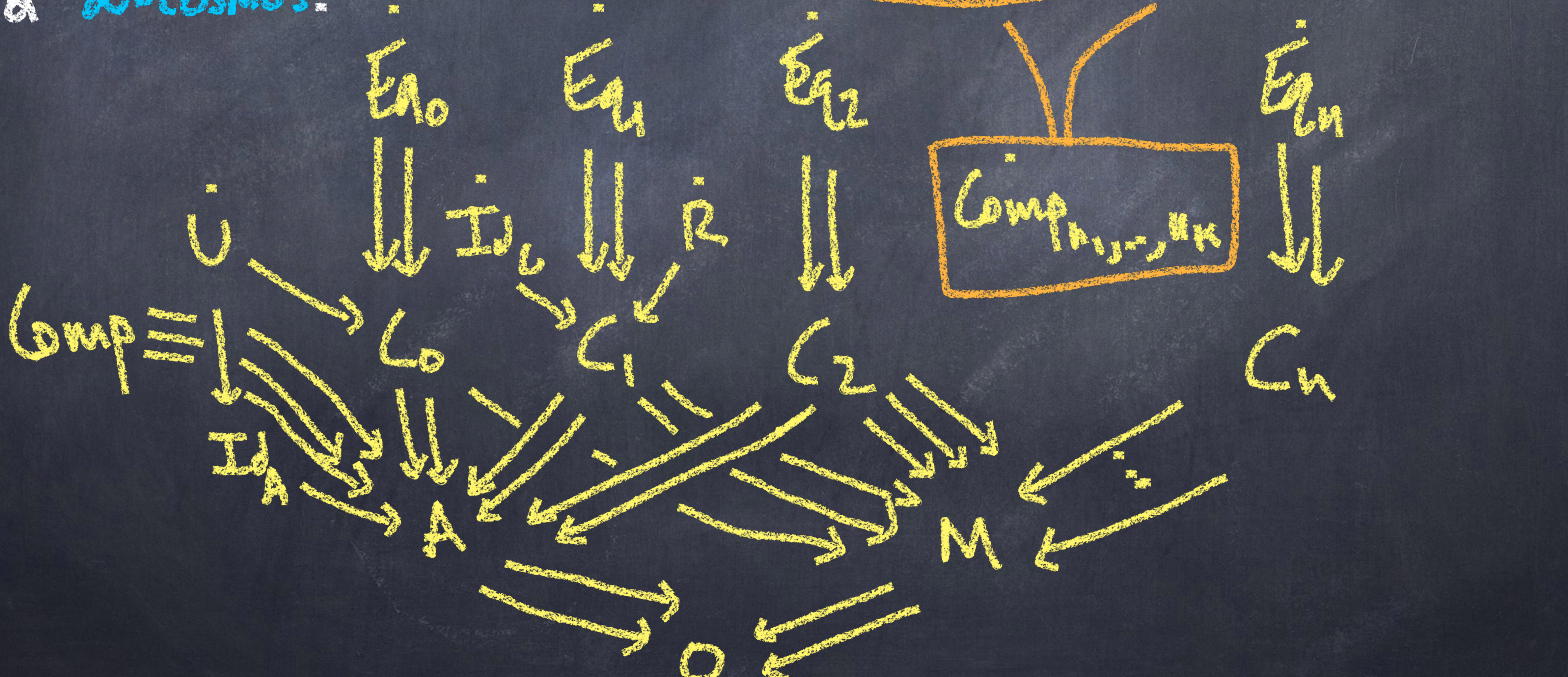
$$\forall x,y\in O,\ \exists f\in A(x,y) \wedge \exists g\in A(y,x),\ g\circ f=id_x \wedge f\circ g=id_y$$

"Any two objects are isomorphic."

The formal language for ∞-category theory has the signature:

Theorem (Riehl–Verity) Statements in the formal language for ∞-categories are invariant under change of ∞-cosmos.





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REFERENCE:

Elements of ∞-Category Theory, Emily Riehl + Dominic Verity

Cambridge University Press 2021, www.math.jhu.edu/~eriehl/elements.pdf