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Categorifying cardinal arithmetic

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Plan

Goal: prove $a \times (b + c) = (a \times b) + (a \times c)$ for any natural numbers a, b, and c.

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- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof

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- Step 1: categorification
- Step 2: the Yoneda lemma
- Step 3: representability
- Step 4: the proof
- Epilogue: what was the point of that?

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Step 1: categorification

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Natural numbers a, b, and c encode the sizes of finite sets A, B, and C.

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A: $a = b$ if and only if A and B are isomorphic, which means there exist functions $f\colon A \to B$ and $g\colon B \to A$ that are inverses in the sense that $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. In this case, we write $A \cong B$.

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Categorification: the truth behind $a = b$ is $A \cong B$.

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Q: What is the deeper meaning of the symbols "$+$" and "$\times$"?

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A: The disjoint union $B + C$ is a set with $b + c$ elements.

$$B = \left\{ \begin{matrix} \sharp \\ \flat \\ \natural \end{matrix} \right\}, \qquad C = \left\{ \begin{matrix} \spadesuit & \heartsuit \\ \diamondsuit & \clubsuit \end{matrix} \right\}, \qquad B + C = \left\{ \begin{matrix} \sharp & \flat & \spadesuit & \heartsuit \\ \natural & \diamondsuit & \clubsuit & \end{matrix} \right\}$$

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A: The cartesian product $A \times B$ is a set with $a \times b$ elements.

$$A = \left\{ \ast \quad \star \right\}, \qquad B = \left\{ \begin{matrix} \sharp \\ \flat \\ \natural \end{matrix} \right\}, \qquad A \times B = \left\{ \begin{matrix} (\ast, \sharp) & (\star, \sharp) \\ (\ast, \flat) & (\star, \flat) \\ (\ast, \natural) & (\star, \natural) \end{matrix} \right\}$$

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A: It means that the sets $A \times (B + C)$ and $(A \times B) + (A \times C)$ are isomorphic!

$$A \times (B + C) \cong (A \times B) + (A \times C)$$

Summary of Step 1

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$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

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Step 2: the Yoneda lemma

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$$
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\mathrm{Fun}(A,A) \arrow[r, phantom, "\cong"] & \mathrm{Fun}(B,A) & & \mathrm{Fun}(A,B) \arrow[r, phantom, "\cong"] & \mathrm{Fun}(B,B) \\
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$$
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\mathsf{Fun}(A, A) & \cong & \mathsf{Fun}(B, A) \\
\cup\!\!\!\!\vert & & \cup\!\!\!\!\vert \\
\mathrm{id}_A & \longmapsto & g
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{Fun}(A, B) & \cong & \mathsf{Fun}(B, B) \\
\cup\!\!\!\!\vert & & \cup\!\!\!\!\vert \\
f & \longleftarrow\!\!\shortmid & \mathrm{id}_B
\end{array}
$$

to define functions $g\colon B \to A$ and $f\colon A \to B$. By naturality:

$$
\begin{tikzcd}
\mathrm{id}_A \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & g \arrow[ddd, mapsto] \\
& \mathsf{Fun}(A, A) \arrow[ul, phantom, "\ni"] \arrow[r, phantom, "\cong"] \arrow[d, "f\circ -"'] & \mathsf{Fun}(B, A) \arrow[ur, phantom, "\ni"] \arrow[d, "f\circ -"] & \\
& \mathsf{Fun}(A, B) \arrow[dl, phantom, "\ni"] \arrow[r, phantom, "\cong"] & \mathsf{Fun}(B, B) \arrow[dr, phantom, "\ni"] & \\
f \arrow[rrr, mapsto] & & & \mathrm{id}_B = f \circ g
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to define functions $g\colon B \to A$ and $f\colon A \to B$. By naturality:

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f \arrow[rrr, mapsto] \arrow[ur, phantom, sloped, "\in"] & & & \mathrm{id}_B = f\circ g \arrow[ul, phantom, sloped, "\ni"]
\end{tikzcd}$$

and similarly $g \circ f = \mathrm{id}_A$. $\square$

Summary of Steps 1 and 2

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$\rightsquigarrow$ we'll instead show that $A \times (B + C) \cong (A \times B) + (A \times C)$.

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By the Yoneda lemma:

Step 2 summary: To prove $A \times (B + C) \cong (A \times B) + (A \times C)$
$\rightsquigarrow$ we'll instead define a "natural" isomorphism
$$\mathrm{Fun}(A \times (B + C), X) \cong \mathrm{Fun}((A \times B) + (A \times C), X).$$

3

Step 3: representability

The universal property of the disjoint union

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A: For each $b \in B$, we need to specify $f(b) \in X$, and for each $c \in C$, we need to specify $f(c) \in X$. So the function $f\colon B + C \to X$ is determined by *two* functions $f_B\colon B \to X$ and $f_C\colon C \to X$.

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A: For each $b \in B$, we need to specify $f(b) \in X$, and for each $c \in C$, we need to specify $f(c) \in X$. So the function $f\colon B + C \to X$ is determined by *two* functions $f_B\colon B \to X$ and $f_C\colon C \to X$.

By "pairing"

$$\begin{array}{ccc} \mathrm{Fun}(B + C, X) & \cong & \mathrm{Fun}(B, X) \times \mathrm{Fun}(C, X) \\ \uplus & & \uplus \\ f & \leftrightsquigarrow & (f_B, f_C) \end{array}$$

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A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a, b) \in X$.

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A: For each $b \in B$ and $a \in A$, we need to specify an element $f(a, b) \in X$. Thus, for each $b \in B$, we need to specify a function $f(-, b)\colon A \to X$ sending a to $f(a, b)$.

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4

Step 4: the proof

The proof

Theorem. For any natural numbers a, b, and c,

$$a \times (b + c) = (a \times b) + (a \times c).$$

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□

5

Epilogue: what was the point of that?

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Proof: The one we just gave.

Exercise: Find a similar proof for other identities of cardinal arithmetic:

$$\alpha^{\beta+\gamma} = \alpha^{\beta} \times \alpha^{\gamma} \quad \text{and} \quad (\alpha^{\beta})^{\gamma} = \alpha^{\beta\times\gamma} = (\alpha^{\gamma})^{\beta}.$$

Generalization to other mathematical contexts

In the discussion of representability or the Yoneda lemma, we didn't need A, B, and C to be sets at all!

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- For vector spaces U, V, W,
$$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W).$$
- For nice topological spaces X, Y, Z,
$$X \times (Y \sqcup Z) = (X \times Y) \sqcup (X \times Z).$$
- For abelian groups A, B, C,
$$A \otimes_{\mathbb{Z}} (B \oplus C) \cong (A \otimes_{\mathbb{Z}} B) \oplus (A \otimes_{\mathbb{Z}} C).$$

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Thank you!