Emily Riehl

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Collaborative Formalizations of ∞-Category Theory

UCLouvain, with the support of the Hoover Foundation

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Strategy II (synthetic). Axiomatize the $(\infty, 2)$-category of ∞-categories using the notion of ∞-cosmos or something similar. State and prove theorems about ∞-categories in the axiomatic language of an ∞-cosmos and its quotient 2-category. To show that this theory is non-vacuous, prove the quasi-categories define an ∞-cosmos (and formalize other examples, as desired).

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Strategy III (extra synthetic). Avoid the technicalities of set-based models by developing the theory of ∞-categories synthetically, in a domain-specific type theory. In simplicial homotopy type theory, an ∞-category can be defined to be a type with unique binary composition of arrows in which paths are equivalent to isomorphisms. Formalization then requires a bespoke proof assistant such as `Rzk`.

Plan

1. Formalizing synthetic ∞-category theory via ∞-cosmoi in `Lean`

2. Formalizing synthetic ∞-category theory in simplicial HoTT in `Rzk`

1

Formalizing synthetic ∞-category theory via ∞-cosmoi in `Lean`

Quasi-categories in Lean

Lean's mathematics library `Mathlib` knows the definition of a quasi-category, thanks to Johan Commelin:

```
/-- A simplicial set `S` is a *quasicategory* if it satisfies the following horn-filling condition:
for every `n : ℕ` and `0 < i < n`,
every map of simplicial sets `σ₀ : Λ[n, i] → S` can be extended to a map `σ : Δ[n] → S`.

[Kerodon, 003A] -/
class Quasicategory (S : SSet) : Prop where
  hornFilling' : ∀ ⦃n : ℕ⦄ ⦃i : Fin (n+3)⦄ (σ₀ : Λ[n+2, i] ⟶ S)
    (_h0 : 0 < i) (_hn : i < Fin.last (n+2)),
      ∃ σ : Δ[n+2] ⟶ S, σ₀ = hornInclusion (n+2) i ≫ σ
```

Here a simplicial set S is a quasi-category if it satisfies a certain property: namely if any inner horn σ_0 in S can be extended to a simplex σ.

$$\begin{tikzcd}
\Lambda[n+2, i] \arrow[r, "\sigma_0"] \arrow[d, hook, "\text{hornInclusion } (n+2)\ i"'] & S \\
\Delta[n+2] \arrow[ur, dashed, "\sigma"']
\end{tikzcd}$$

∞-cosmoi in Lean?

`Mathlib` also knows the definition of an enriched category. Thus it should be feasible to formalize the following definition:

1.2.1. Definition (∞-cosmos). An ∞-**cosmos** $\mathcal{K}$ is a category that is enriched over quasi-categories,[13] meaning in particular that

- its morphisms $f \colon A \to B$ define the vertices of a quasi-category denoted $\mathsf{Fun}(A, B)$ and referred to as a **functor space**,

that is also equipped with a specified collection of maps that we call **isofibrations** and denote by "$\twoheadrightarrow$" satisfying the following two axioms:

(i) (completeness) The quasi-categorically enriched category $\mathcal{K}$ possesses a terminal object, small products, pullbacks of isofibrations, limits of countable towers of isofibrations, and cotensors with simplicial sets, each of these limit notions satisfying a universal property that is enriched over simplicial sets.[14]

(ii) (isofibrations) The isofibrations contain all isomorphisms and any map whose codomain is the terminal object; are closed under composition, product, pullback, forming inverse limits of towers, and Leibniz cotensors with monomorphisms of simplicial sets; and have the property that if $f \colon A \twoheadrightarrow B$ is an isofibration and X is any object then $\mathsf{Fun}(X, A) \twoheadrightarrow \mathsf{Fun}(X, B)$ is an isofibration of quasi-categories.

The ∞-cosmos project

Last month, Mario Carneiro, Dominic Verity, and I launched the ∞-cosmos project:

∞-Cosmos

A project to formalize ∞-cosmoi in Lean.

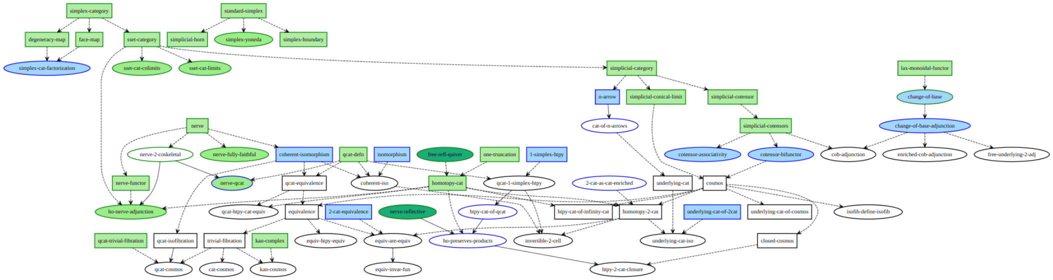
Blueprint (web) | Blueprint (pdf) | Documentation | GitHub

Useful links:

- Zulip chat for Lean for coordination
- Blueprint
- Blueprint as pdf
- Dependency graph
- Doc pages for this repository

A blueprint for the formalization project

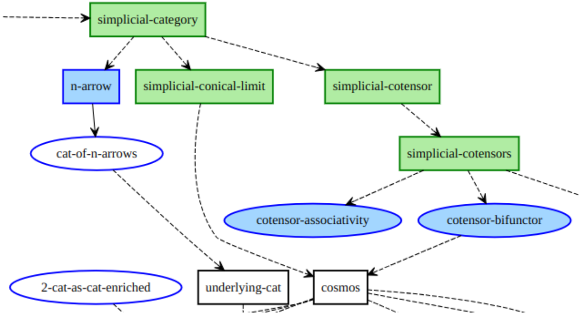
Pietro Monticone and Patrick Massot helped us set up a blueprint (and website) to organize the workflow:



There is a lot of work that remains to be done!

Towards the definition of an ∞-cosmos

The definition of an ∞-cosmos requires the notion of a simplicially enriched category and also the notions of simplicially enriched limits.



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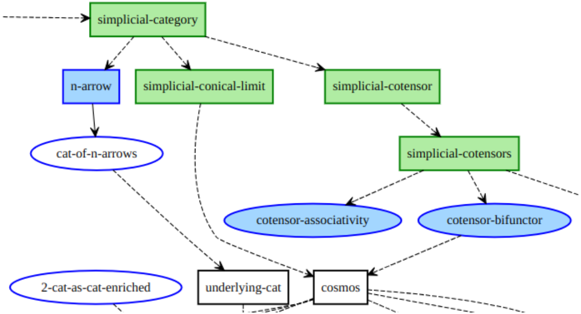

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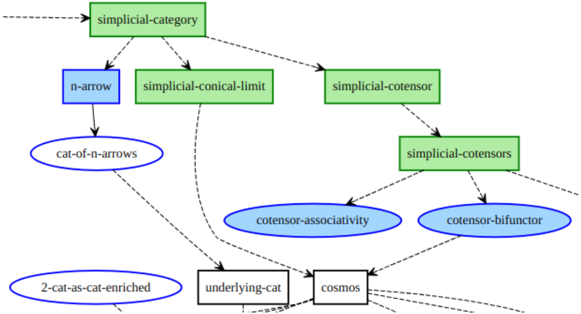

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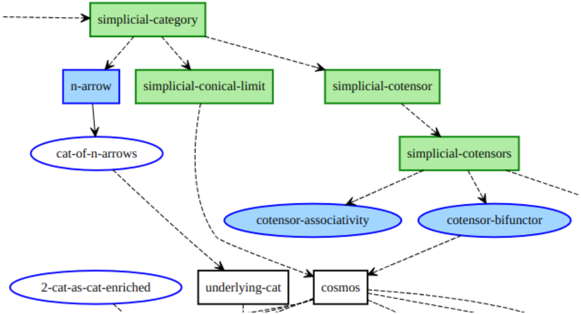

- Conical limits and cotensors have been formalized since the start of this project.
- Work-in-progress will formalize the bifunctor defined by the simplicial cotensor.
- Once this is done, the definition of an ∞-cosmos will be formalizable.

A formalization target

The blueprint describes a lemma that remains to be formalized:

Lemma 1.2.29

1. The functor $\mathsf{h}\colon s\mathcal{S}et \to \mathcal{C}at$ preserves finite products.
2. The functor $\mathsf{h}\colon \mathcal{QC}at \to \mathcal{C}at$ preserves small products.

Proof ▾

For the first statement, preservation of the terminal object is by direct calculation. By Proposition 1.2.25, preservation of binary products is equivalent to the statement that the canonical map $N(\mathcal{D}^{\mathcal{C}}) \to N(\mathcal{D})^{N\mathcal{C}}$ involving nerves of categories is an isomorphism. On n-simplices, this is defined by uncurrying, which is bijection since $\mathcal{C}at$ is cartesian closed.

For the second statement, we have a canonical comparison functor from the homotopy category of the products to the product of the homotopy categories. It follows from Lemma 1.2.28 that this is an isomorphism on underlying quivers, which suffices. □

Contributors to the ∞-cosmos project

So far formalizations (and preliminary mathematical work) have been contributed by:

Dagur Asgeirsson, Mario Carneiro, Johan Commelin, Jack McKoen, Pietro Monticone, Emily Riehl, Joël Riou, Joseph Tooby-Smith, Adam Topaz, and Dominic Verity.

Anyone is welcome to join us!

emilyriehl.github.io/infinity-cosmos

2

Formalizing synthetic ∞-category theory in simplicial HoTT in `Rzk`

∞-categories in set theory

Essentially, ∞-categories are 1-categories in which all the sets have been replaced by ∞-groupoids aka homotopy types:

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- categories have sets of objects, ∞-categories have ∞-groupoids of objects, and
- categories have hom-sets, ∞-categories have ∞-groupoidal mapping spaces.

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While the axioms that turn a directed graph into a category are expressed in the language of set theory — a category has a composition function satisfying axioms expressed in first-order logic with equality — composition in an ∞-category, as a morphism between ∞-groupoids, isn't a "function" in the traditional sense (since homotopy types do not have underlying sets of points).

This is why ∞-categories are so difficult to model within set theory.

Could ∞-category theory be taught to undergraduates?

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Could ∞-Category Theory Be Taught to Undergraduates?

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1. The Algebra of Paths

It is natural to probe a suitably nice topological space X by means of its *paths*, the continuous functions from the standard unit interval $I \coloneqq [0,1] \subset \mathbb{R}$ to X. But what structure do the paths in X form?

To start, the paths form the edges of a directed graph whose vertices are the points of X: a path $p : I \to X$ defines an arrow from the point $p(0)$ to the point $p(1)$. Moreover, this graph is *reflexive*, with the constant path refl_x at each point $x \in X$ defining a distinguished endoarrow.

Can this reflexive directed graph be given the structure of a category? To do so, it is natural to define the composite of a path p from x to y and a path q from y to z by gluing together these continuous maps—i.e., by concatenating the paths—and then by reparametrizing via the homeomorphism $I \cong I \cup_{1=0} I$ that traverses each path at double speed:

$$\begin{tikzcd} I \arrow[r, "\cong"] \arrow[rr, bend right, "p*q"'] & I \cup_{1=0} I \arrow[r, "p\cup q"] & X \end{tikzcd} \tag{1.1}$$

But the composition operation $*$ fails to be associative or unital. In general, given a path r from z to w, the

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The traditional foundations of mathematics are not really suitable for "higher mathematics" such as ∞-category theory, where the basic objects are built out of higher-dimensional types instead of mere sets. However, there are proposals for new foundations for mathematics that are closer to mathematician's core intuitions, based on Martin-Löf's dependent type theory such as

- homotopy type theory,
- higher observational type theory, and the
- simplicial type theory, that we use here.

∞-categories in simplicial homotopy type theory

The identity type family gives each type the structure of an ∞-groupoid: each type A has a family of identity types over $x, y : A$ whose terms $p : x =_A y$ are called paths.

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Emily Riehl and Michael Shulman, A type theory for synthetic ∞-categories,
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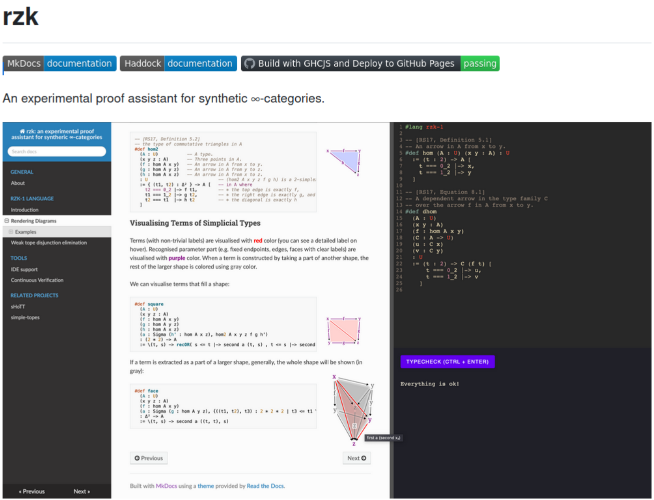
With more of the work being done by the foundation system, perhaps someday ∞-category theory will be easy enough to teach to undergraduates?

An experimental proof assistant Rzk for ∞-category theory



The proof assistant Rzk was written by Nikolai Kudasov:

About this project

This project has started with the idea of bringing Riehl and Shulman's 2017 paper [1] to "life" by implementing a proof assistant based on their type theory with shapes. Currently an early prototype with an online playground is available. The current implementation is capable of checking various formalisations. Perhaps, the largest formalisations are available in two related projects: https://github.com/fizruk/sHoTT and https://github.com/emilyriehl/yoneda. `sHoTT` project (originally a fork of the yoneda project) aims to cover more formalisations in simplicial HoTT and ∞-categories, while `yoneda` project aims to compare different formalisations of the Yoneda lemma.

Internally, `rzk` uses a version of second-order abstract syntax allowing relatively straightforward handling of binders (such as lambda abstraction). In the future, `rzk` aims to support dependent type inference relying on E-unification for second-order abstract syntax [2]. Using such representation is motivated by automatic handling of binders and easily automated boilerplate code. The idea is that this should keep the implementation of `rzk` relatively small and less error-prone than some of the existing approaches to implementation of dependent type checkers.

An important part of `rzk` is a tope layer solver, which is essentially a theorem prover for a part of the type theory. A related project, dedicated just to that part is available at https://github.com/fizruk/simple-topes. `simple-topes` supports used-defined cubes, topes, and tope layer axioms. Once stable, `simple-topes` will be merged into `rzk`, expanding the proof assistant to the type theory with shapes, allowing formalisations for (variants of) cubical, globular, and other geometric versions of HoTT.

rzk-lang.github.io/rzk

A formalized proof of the ∞-categorical Yoneda lemma

Our initial aim was to write a formalized proof of the ∞-categorical Yoneda lemma.

github.com/emilyriehl/yoneda or emilyriehl.github.io/yoneda/

- proof from Emily Riehl & Mike Shulman, A type theory for synthetic ∞-categories, Higher Structures 2017.
- formalizations written by Nikolai Kudasov, Emily Riehl, Jonathan Weinberger.
- completed March 12 – April 17, 2023

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Another objective is to compare ∞-category theory in simplicial type theory with ordinary category theory in traditional foundations. Thus,

- We've included a formalization of the 1-categorical Yoneda lemma in `Lean` by Sina Hazratpour as part of an Introduction to Proofs course at Johns Hopkins.
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More recently, we've professionalized our library, implementing a style guide suggested by Fredrik Bakke, and invited new contributors to a broader project of formalizing synthetic ∞-category theory:

github.com/rzk-lang/sHoTT or rzk-lang.github.io/sHoTT

Contributors to the simplicial HoTT library

So far formalizations (and work on the proof assistant `Rzk`) have been contributed by:

Abdelrahman Aly Abounegm, Fredrik Bakke, César Bardomiano Martínez, Jonathan Campbell, Robin Carlier, Theofanis Chatzidiamantis-Christoforidis, Aras Ergus, Matthias Hutzler, Nikolai Kudasov, Kenji Maillard, David Martínez Carpena, Stiéphen Pradal, Nima Rasekh, Emily Riehl, Florrie Verity, Tashi Walde, and Jonathan Weinberger.

Anyone is welcome to join us!

rzk-lang.github.io/sHoTT

References

Papers:

- Emily Riehl, Could ∞-category theory be taught to undergraduates?, Notices of the AMS 70(5):727–736, May 2023; arXiv:2302.07855
- Nikolai Kudasov, Emily Riehl, Jonathan Weinberger, Formalizing the ∞-categorical Yoneda lemma, CPP 2024: 274–290; arXiv:2309.08340

Formalization:

- Johan Commelin, Kim Morrison, Joël Riou, Adam Topaz, a nascent theory of quasi-categories in Mathlib, AlgebraicTopology/SimplicialSet/Quasicategory.lean
- Mario Carneiro, Emily Riehl, and Dominic Verity, a blueprint of the model-independent theory, emilyriehl.github.io/infinity-cosmos
- Nikolai Kudasev et al, synthetic ∞-categories in simplicial homotopy type theory, rzk-lang.github.io/sHoTT/

Merci!