Contractibility as Uniqueness

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UCLA Distinguished Lecture Series
An analogy

contractibility :: uniqueness

1. Contractibility as Uniqueness

2. Categorifying Uniqueness

3. $\infty$-Categorifying Uniqueness
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Contractibility as Uniqueness
The algebra of paths

The standard technique used to distinguish your favorite space $A$ from other spaces is to compute an algebraic invariant of the space.

The “algebra of paths” of a space is described in increasing precision by:

- the fundamental group $\pi_1(A, x)$ of loops in $A$ based at $x$ up to homotopy
- the fundamental groupoid $\pi_1 A$ of paths in $A$ up to homotopy
- the fundamental $\infty$-groupoid $\pi_\infty A$ of paths in $A$

$\pi_\infty A$ has:

- points of $A$ as objects
- paths of $A$ as 1-arrows
- paths between paths in $A$ as 2-arrows
- paths between paths between paths in $A$ as 3-arrows, and so on ...
Witnesses to composition

**Q:** How do we define the composite of two paths?  
**A:** We don’t!

Instead of a composition operation, composites of paths are witnessed by higher paths.

![Diagram](image)

**Q:** How unique is path composition?  
**Partial A:** Unique enough for associativity.

Given composable paths $f$, $g$, $h$ and specified higher paths $\alpha$, $\beta$, $\gamma$ witnessing composition relations, these higher paths compose. More precisely, a 3-arrow expresses a coherence between compositions witnessed by 2-arrows.
Theorem. The space of composites of two paths \( f \) and \( g \) in \( A \) is contractible.

Proof: The space of composites of paths \( f \) and \( g \) in \( A \) is defined by the pullback:

\[
\begin{array}{cccc}
S^{n-1} & \rightarrow & \text{Comp}(f, g) & \rightarrow & A^\Delta \\
\downarrow & & \downarrow & & \downarrow \text{restrict} \\
D^n & \rightarrow & * & \rightarrow & A^\Lambda \\
\end{array}
\]

\[
\begin{array}{cccc}
S^{n-1} \times \Delta \cup_{S^{n-1} \times \Lambda} D^n \times \Lambda & \rightarrow & A \\
D^n \times \Delta & \rightarrow & \\
\end{array}
\]

A space is \textit{contractible} just when any sphere \( S^{n-1} \) can be filled to a disk \( D^n \) for \( n \geq 1 \). This filling problem transposes to an extension problem, and the extension exists since the inclusion admits a continuous deformation retract.
Summary

- In a group(oid), composable arrows have a unique composite.
- In a $\infty$-group(oid), composable arrows have a contractible space of composites.

The analogy

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ordinary mathematics :: higher mathematics
  uniqueness     :: contractibility
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can be made even tighter.

Aim: Express the classical notion of uniqueness more categorically.
Categorifying Uniqueness
Uniqueness

To say $C$ has a unique element means

$$\exists x \in C, \forall y \in C, x = y$$

Here “$x = y$” is a predicate — a mathematical statement that is either true or false, depending on two free variables $x, y \in C$.

In proof-relevant mathematics, we interpret “$x = y$” as the set of all proofs that $x$ equals $y$ (which is empty if $x$ and $y$ are not equal).

Then we can form the set $\sum_{x \in C} \prod_{y \in C} x = y$

inspired by a notational analogy with the sentence $\exists x \in C, \forall y \in C, x = y$.

The set $\sum_{x \in C} \prod_{y \in C} x = y$ is also a set of proofs — but proofs of what?
Proofs of uniqueness

The set $\sum_{x \in C} \prod_{y \in C} x = y$ is also a set of proofs — but proofs of what?

An element of $\sum_{x \in C} \prod_{y \in C} x = y$ is

- the choice of some element $c \in C$
- together with a proof, for all $z \in C$, that $c$ equals $z$.

Thus $\sum_{x \in C} \prod_{y \in C} x = y$ is the set of proofs of the sentence $\exists x \in C, \forall y \in C, x = y$

asserting that $C$ has a unique element.

It remains to explain the analogy:
Digression: quantifiers as adjoints

A set function $f : S \rightarrow T$ induces order-preserving functions between their powersets:

- $\exists_f$ is direct image: $A \subset S \mapsto \{t \in T \mid \exists s \in S, f(s) = t \land s \in A\} \subset T$
- $\forall_f$ is pushforward: $A \subset S \mapsto \{t \in T \mid \forall s \in S, f(s) = t \Rightarrow s \in A\} \subset T$
- $\Delta_f$ is inverse image: $B \subset T \mapsto \{s \in S \mid f(s) \in B\} \subset S$

For the unique function $! : S \rightarrow \ast$ these reduce to $P(S) \leftarrow \Delta \rightarrow P(\ast) = \{\ast, \emptyset\}$

The set $P(S) = \{A \subset S\}$ can be identified with the set of predicates $p(s)$ with one free variable $s \in S$ — the corresponding subset is $\{s \in S \mid p(s) \text{ is true}\}$. If we interpret the two elements of $P(\ast)$ by $\ast =: \text{"true"}$ and $\emptyset =: \text{"false"}$ then

- $\exists$ is the function that sends the predicate $p(s)$ to the sentence $\exists s \in S, p(s)$
- $\forall$ is the function that sends the predicate $p(s)$ to the sentence $\forall s \in S, p(s)$
Digression: locally cartesian closed categories

For any function \( f : S \rightarrow T \) there are functors:

\[ P(S) \xleftarrow{\Delta_f} P(T) \]

\( \exists_f \) is inverse image: \( B \subset T \mapsto \{ s \in S \mid f(s) \in B \} \subset S \)

\( \forall_f \) is pushforward: \( A \subset S \mapsto \{ t \in T \mid \forall s \in S, f(s) = t \Rightarrow s \in A \} \subset T \)

\( \exists_f \) is direct image: \( A \subset S \mapsto \{ t \in T \mid \exists s \in S, f(s) = t \wedge s \in A \} \subset T \)

\( \forall_f \) is pushforward: \( A \subset S \mapsto \{ t \in T \mid \forall s \in S, f(s) = t \Rightarrow s \in A \} \subset T \)

In proof-relevant mathematics, it is natural to replace the poset \( P(S) \) by the category \( \text{Set}_S \) of \( S \)-indexed sets. An object \( \{P(s)\}_{s \in S} \) is a family of sets where \( P(s) \) can be thought of as the set of proofs of some predicate \( p(s) \) on \( s \in S \).

\[ \text{Set}_S \xleftarrow{\Delta_f} \text{Set}_T \]

\( \sum_f \) is sum: \( \{P(s)\}_{s \in S} \mapsto \{ \sum_{s \in f^{-1}(t)} P(s) \}_{t \in T} \)

\( \prod_f \) is product: \( \{P(s)\}_{s \in S} \mapsto \{ \prod_{s \in f^{-1}(t)} P(s) \}_{t \in T} \)
The triple of adjoint functors

\[ \text{Set}_S \xleftarrow{\sum f} \text{Set}_T \xrightarrow{\prod f} \]

\[ \Delta_f \text{ is substitution: } \{Q(t)\}_{t \in T} \mapsto \{Q(f(s))\}_{s \in S} \]

\[ \sum f \text{ is sum: } \{P(s)\}_{s \in S} \mapsto \{\sum_{s \in f^{-1}(t)} P(s)\}_{t \in T} \]

\[ \prod_f \text{ is product: } \{P(s)\}_{s \in S} \mapsto \{\prod_{s \in f^{-1}(t)} P(s)\}_{t \in T} \]

gives a more formal way to understand the set

\[ \sum_{x \in C} \prod_{y \in C} x = y \]

\[ \implies \text{The set of proofs “}x = y\text{” defines an indexed set } \{x = y\}_{x,y \in C} \in \text{Set}_{/C \times C} \]

\[ \implies \text{Product along the projection } \pi_1 : C \times C \to C \text{ gives } \{\prod_{y \in C} x = y\}_{x \in C} \in \text{Set}_C \]

\[ \implies \text{Sum along } ! : C \to * \text{ gives the set } \sum_{x \in C} \prod_{y \in C} x = y \in \text{Set}_{/*} = \text{Set} \]
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∞-Categorifying Uniqueness
A convenient category of spaces

Now replace Set by a convenient category Space of spaces and continuous maps.

A family of spaces \( \{E_b\}_{b \in B} \in \text{Space}/B \) is a continuous map \( \pi : E \to B \), where the space \( E_b \) is the fiber over a point \( b \in B \) in the base space while the total space \( E \simeq \sum_{b \in B} E_b \).

Any continuous \( f : S \to T \) gives rise to an adjoint triple:

\[
\begin{array}{ccc}
\text{Space}/S & \xleftarrow{\Delta_f} & \text{Space}/T \\
\sum_f &\perp & \Pi_f \\
\end{array}
\]

\( \Delta_f \) is pullback
\( \sum_f \) is composition
\( \Pi_f \) is pushforward
Identifications as paths

Q: For a space $C$, how to interpret the family of spaces $\{x = y\}_{x,y \in C} \in \mathit{Space}/C \times C$?

First guess: $\Delta \in \mathit{Space}/C \times C$ — but a better choice is the path space $C' \in \mathit{Space}/C \times C$.

New idea:

A point $p \in x = y$ is a path from $x$ to $y$ in $C$, providing a proof that $x$ equals $y$. 
A space of proofs

What is a point in the space \( \sum_{x \in C} \prod_{y \in C} x = y \)?

The functor \( \sum_B : \text{Space}_B \to \text{Space} \) takes \( \{E_b\}_{b \in B} \) to the total space \( \sum_{b \in B} E_b \).

\[ \leadsto \text{a point in } \sum_{b \in B} E_b \text{ is a pair } (a, e_a) \text{ of a point } a \in B \text{ and a point } e_a \in E_a \]

The functor \( \prod_B : \text{Space}_B \to \text{Space} \) takes \( \{E_b\}_{b \in B} \) to the space of sections \( \prod_{b \in B} E_b \).

\[ \leadsto \text{a point in } \prod_{b \in B} E_b \text{ is section } s : B \to \sum_{b \in B} E_b \text{ of the projection to } B \]

- So a point in \( \sum_{x \in C} \prod_{y \in C} x = y \) is a pair \((c, h)\) where \( c \in C \) and \( h \in \prod_{y \in C} c = y \).
- The point \( h \in \prod_{y \in C} c = y \) is a section \( h : C \to \sum_{y \in C} c = y \) to the projection.

Together \((c, h) \in \sum_{x \in C} \prod_{y \in C} x = y\) defines:

- a center of contraction \( c \) and
- a contracting homotopy \( h \),

proving that the space \( C \) is contractible!
Contractibility as uniqueness

In summary, a point in the set

$$\sum_{x \in C} \prod_{y \in C} x = y$$

is a proof that $C$ is unique, while a point in the space

$$\sum_{x \in C} \prod_{y \in C} x = y$$

is a proof that $C$ is contractible.

Next time: If identifications $p \in x = y$ are paths, which may carry data, what strategies exist to prove a theorem involving a hypothesis of the form $x = y$? We’ll introduce one powerful proof technique: the principle of path induction in homotopy type theory.

Thank you!