A formal category theory for ∞-categories

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defn (Lawvere–Tierney) An elementary 1-topos is

(0) a cartesian closed 1-category that has

(1) finite limits and

(2) a subobject classifier.

defn (Weber, after Street) A 2-topos is

(0) a cartesian closed 2-category that has

(1) finite limits,

(γ) a duality involution, and

(2) a classifying left fibration.

Fix two Grothendieck universes $U < U'$.

Ex The 1-category **Set** of U-small sets is a **1-topos**.

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \top} \\ A & \underset{\chi_S}{\longrightarrow} & \Omega \end{array}$$

$\Omega = \{\top, \bot\}$ classifies subobjects

Ex The 2-category **CAT** of U'-small categories is a **2-topos**.

$$\begin{array}{ccc} \int F & \longrightarrow & Set_* \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle U} \\ A & \underset{F}{\longrightarrow} & Set \end{array}$$

classifies **left fibrations** with U-small fibers
(discrete opfibrations)

Main result (R-Verity) A weakened version of these 2-topos axioms are satisfied by $\mathcal{CAT}_\infty$, the 2-category of ∞-categories.

Here and elsewhere ∞-category is shorthand for (∞,1)-category, a category weakly enriched in ∞-groupoids/homotopy types.

PLAN

(0) a cartesian closed 2-category $\mathcal{CAT}_\infty$

(1) finite limits in $\mathcal{CAT}_\infty$

(γ) a duality involution on $\mathcal{CAT}_\infty$

(2) a classifying left fibration for $\mathcal{CAT}_\infty$

Part 0: a cartesian closed 2-category $\mathcal{CAT}_\infty$

We can use various models of ∞-categories to define the 2-category $\mathcal{CAT}_\infty$ and the results will be biequivalent.

Theorem (Joyal, Rezk, Joyal–Tierney, Verity) The 1-categories of quasi-categories, complete Segal spaces, Segal categories, and 1-complicial sets are cartesian closed.

⇝ For $\mathcal{K}$ = qCat, CSS, Segal, or 1-Comp, $\mathcal{K}$ is a $\mathcal{K}$-category:

$$C^{A\times B} \cong (C^B)^A \in \mathcal{K}$$

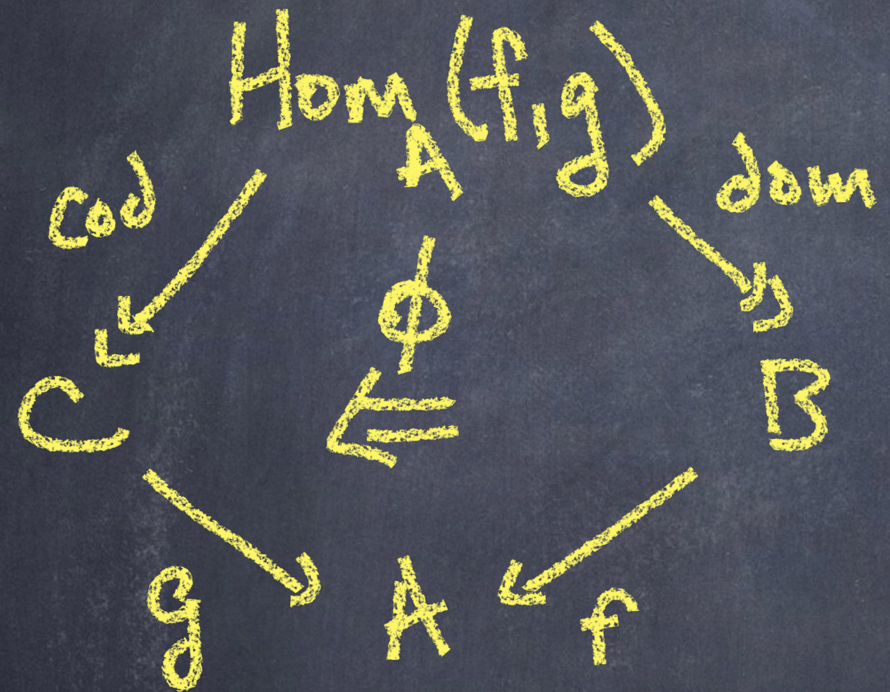
The homotopy category functor $\mathcal{K} \xrightarrow{ho} \mathcal{CAT}$ preserves products ⇝

$\mathcal{K}$ is a cartesian closed 2-category with $\mathrm{fun}(A,B) := ho(B^A)$.

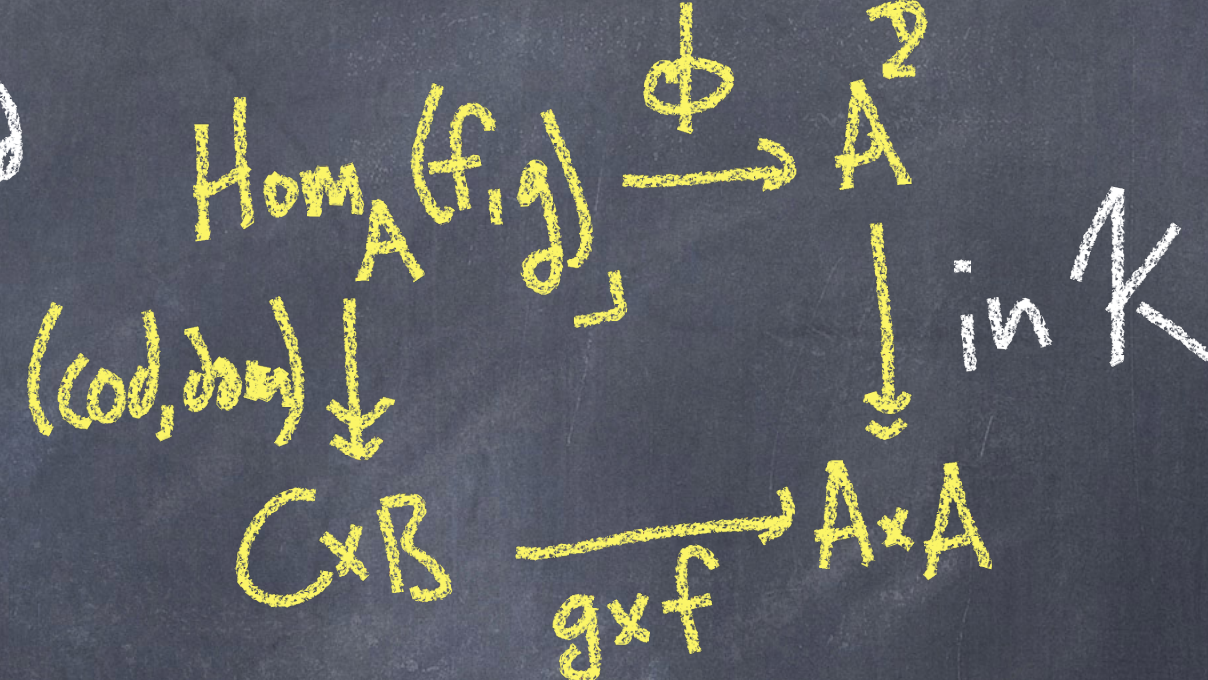
⇝ this defines the 2-category $\mathcal{CAT}_\infty$

Part 1: weak finite limits in $\mathcal{CAT}_\infty$

Prop. Given $C \xrightarrow{g} A \xleftarrow{f} B$ in $\mathcal{CAT}_\infty$ there exists a weak comma object



constructed by



such that

up to iso over $C \times B$.

Cor. $\mathrm{Hom}_A(f,g)$ is unique up to fibered equivalence.

defn. $A \xrightarrow{k} B$ is **fully faithful** iff $A^{\mathbf{2}} \underset{\sim}{\xrightarrow{\ulcorner \mathrm{id}_k \urcorner}} \mathrm{Hom}_A(k,k)$.

defn. A **pointwise right Kan extension** between ∞-categories is

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ & \overset{\nu}{\Leftarrow} & \\ f \searrow & & \swarrow r \\ & C & \end{array}$$

such that

$$\begin{array}{ccc} \mathrm{Hom}_B(b,k) & \twoheadrightarrow & X \\ \downarrow & \overset{\phi}{\Swarrow} & \downarrow b \\ A & \xrightarrow{k} & B \\ & \overset{\nu}{\Leftarrow} & \\ f \searrow & & \swarrow r \\ & C & \end{array}$$

is also a right Kan extension.

Prop. If k is fully faithful, then ν is invertible.

Prop. A **right adjoint** is fully faithful iff the counit is invertible.

Part γ: a duality involution on $\mathcal{CAT}_\infty$

Like $\mathcal{CAT}$, $\mathcal{CAT}_\infty$ has a 2-functor $\mathcal{CAT}_\infty^{co} \xrightarrow{(-)^\circ} \mathcal{CAT}_\infty$ that sends an ∞-category A to its opposite ∞-category A°, such that $(A^\circ)^\circ = A$.

But Weber's notion of duality involution requires more…

To state the full axiom we must introduce:

two-sided discrete fibrations aka modules between ∞-categories.

defn. A functor $E \overset{p}{\twoheadrightarrow} B$ between ∞-categories is a

left fibration iff $E^{\mathbb{2}} \xrightarrow{\ulcorner \mathrm{id}_p \urcorner} \mathrm{Hom}_B(p,B)$ is an equivalence

right fibration iff $E^{\mathbb{2}} \xrightarrow{\ulcorner \mathrm{id}_p \urcorner} \mathrm{Hom}_B(B,p)$ is an equivalence

cocartesian fibration iff $E^{\mathbb{2}} \xrightarrow{\ulcorner \mathrm{id}_p \urcorner} \mathrm{Hom}_B(p,B)$ admits a lari

cartesian fibration iff $E^{\mathbb{2}} \xrightarrow{\ulcorner \mathrm{id}_p \urcorner} \mathrm{Hom}_B(B,p)$ admits a rari

Ex $A^{\mathbb{2}} \xrightarrow{\mathrm{dom}} A$ is a cartesian fibration

$A^{\mathbb{2}} \xrightarrow{\mathrm{cod}} A$ is a cocartesian fibration

Ex For $1 \xrightarrow{a} A$, $\mathrm{Hom}_A(A,a) \xrightarrow{\mathrm{dom}} A$ is a right fibration

$\mathrm{Hom}_A(a,A) \xrightarrow{\mathrm{cod}} A$ is a left fibration

defn. A span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ of ∞-categories defines a **module from A to B** iff

- $A \overset{q}{\twoheadleftarrow} E$ is a cocartesian fibration over B
- $E \overset{p}{\twoheadrightarrow} B$ is a cartesian fibration over A
- the fibers of $E \xrightarrow{(q,p)} A \times B$ are ∞-groupoids

Ex any left fibration $A \overset{q}{\twoheadleftarrow} E \twoheadrightarrow 1$ or any right fibration $1 \twoheadleftarrow E \overset{p}{\twoheadrightarrow} B$

Ex $A \overset{cod}{\twoheadleftarrow} A^{2} \overset{dom}{\twoheadrightarrow} A$

$C \overset{cod}{\twoheadleftarrow} \mathrm{Hom}_A(f,g) \overset{dom}{\twoheadrightarrow} B$ for any $C \xrightarrow{g} A \xleftarrow{f} B$

$1 \twoheadleftarrow \mathrm{Hom}_A(A,a) \overset{dom}{\twoheadrightarrow} A$ for any $1 \xrightarrow{a} A$

$A \overset{cod}{\twoheadleftarrow} \mathrm{Hom}_A(a,A) \twoheadrightarrow 1$

Idea: a module $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ encodes a homotopy coherent diagram $A \times B^{\circ} \to \infty\text{-Gpd}$ ... but so does a left fibration $F \twoheadrightarrow A \times B^{\circ}$ or a right fibration $G \twoheadrightarrow A^{\circ} \times B$

defn (Weber) A duality involution entails an involutive 2-functor $(-)^{\circ}$, contravariant on 2-cells, together with a pseudo-natural equivalence of categories

$$\left\{\begin{array}{c}\text{modules from}\\ A \times B^{\circ} \text{ to } C\end{array}\right\} \simeq \left\{\begin{array}{c}\text{modules from}\\ A \text{ to } B \times C\end{array}\right\}$$

Our next task is to construct this for $\mathcal{CAT}_\infty$.

We work with the quasi-categorical model of ∞-categories.

defn. The quasi-category $A{\scriptstyle 2}A$ of twisted arrows in a quasi-category A has

$$\{\Delta[n] \to A{\scriptstyle 2}A\} \cong \{\Delta[n]^{\circ} \star \Delta[n] \to A\}$$

Consider also $A{\scriptstyle 2}^{\circ}A := (A{\scriptstyle 2}A)^{\circ}$

$$\{\Delta[n] \to A{\scriptstyle 2}^{\circ}A\} \cong \{\Delta[n] \star \Delta[n]^{\circ} \to A\}$$

Prop. $A{\scriptstyle 2}A \xrightarrow{(\mathrm{cod},\mathrm{dom})} A \times A^{\circ}$ is a left fibration.

$A{\scriptstyle 2}^{\circ}A \xrightarrow{(\mathrm{cod},\mathrm{dom})} A^{\circ} \times A$ is a right fibration.

Prop. For all $1 \xrightarrow{a} A$,

$$\begin{array}{ccc} a{\scriptstyle 2}A & \simeq & \mathrm{Hom}_A(a,A) \\ & {\scriptstyle \mathrm{cod}}\searrow \quad \swarrow{\scriptstyle \mathrm{cod}} & \\ & A & \end{array} \quad \text{and} \quad \begin{array}{ccc} A{\scriptstyle 2}^{\circ}a & \simeq & \mathrm{Hom}_A(A,a) \\ & {\scriptstyle \mathrm{dom}}\searrow \quad \swarrow{\scriptstyle \mathrm{dom}} & \\ & A & \end{array}$$

$\leadsto$ $A{\scriptstyle 2}A$ and $A{\scriptstyle 2}^{\circ}A$ are twisted versions of $A^{\mathbf{2}}$.

Theorem. The modules

$$\left(A\times A^{\circ} \leftarrow A\downarrow A \twoheadrightarrow 1 \right) \quad \text{and} \quad \left(1 \leftarrow A\downarrow^{\circ} A \twoheadrightarrow A^{\circ}\times A \right)$$

are duals in the monoidal bicategory of modules:

$$\left(A \leftarrow A\times A\downarrow^{\circ} A \twoheadrightarrow A\times A^{\circ}\times A \right) \underset{A\times A^{\circ}\times A}{\otimes} \left(A\times A^{\circ}\times A \leftarrow A\downarrow A\times A \twoheadrightarrow A \right)$$

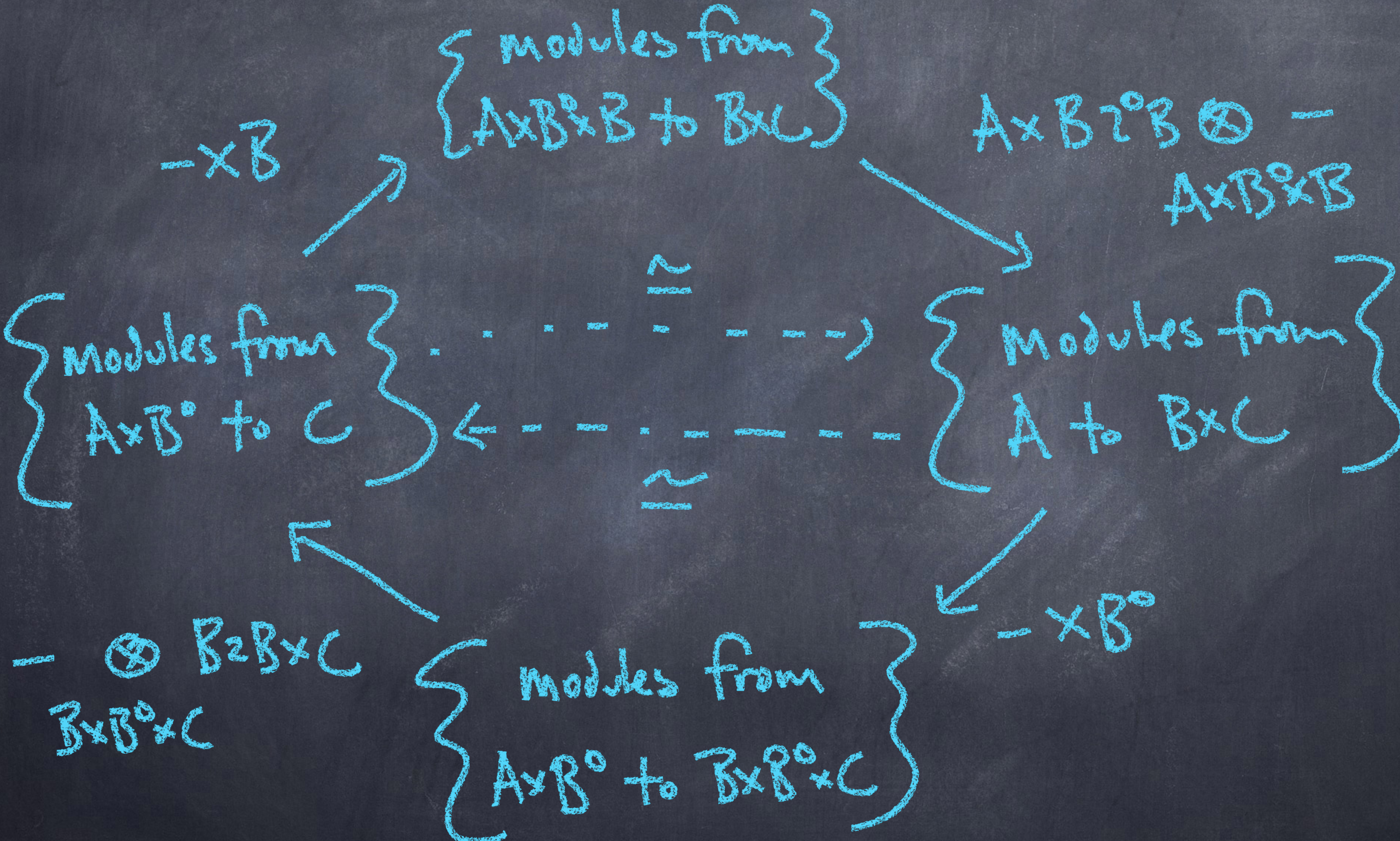
$$\cong$$

$$\left(A \xleftarrow{\text{cod}} A^{2} \xrightarrow{\text{dom}} A \right)$$

as modules from A to A.

Yoneda Lemma $\rightsquigarrow$ A^{2} is the unit for $\otimes_{A}$.

Theorem. The 2-category CAT_∞ has a duality involution



Part 2: a classifying left fibration for $\mathcal{CAT}_\infty$

defn (Weber). A left fibration $S_* \overset{v}{\twoheadrightarrow} S$ is classifying when the functor $\text{fun}(A,S) \longrightarrow \text{leftfib}/_A$ given by

$$\begin{array}{ccc} \int f & \longrightarrow & S_* \\ \downarrow\!\!\!\!\downarrow \lrcorner & & \downarrow\!\!\!\!\downarrow v \\ A & \underset{f}{\longrightarrow} & S \end{array}$$

is fully faithful for all A.

IDEA: Take $S \in \mathcal{CAT}_\infty$ to be the ∞-category of U-small ∞-groupoids and take S_* to be the ∞-category of pointed ∞-groupoids.

Again we work with the quasi-categorical model of ∞-categories.

Strategies for constructing the classifying left fibration $\mathcal{S}_* \xrightarrow{u} \mathcal{S}$

- "unstraightening id": define

  $\mathcal{S}$ = the homotopy coherent nerve of a cartesian closed category of spaces

  $\mathcal{S}_*$ = the slice quasi-category $^{*/}\mathcal{S}$

- via locality of left fibrations: take

  $\{\Delta[n] \to \mathcal{S}\} \simeq$ "{U-small left fibrations over $\Delta[n]$}"

  $\{\Delta[n] \to \mathcal{S}_*\} \simeq$ "{U-small left fibrations over $\Delta[n]$ together with a global section}"

- model-independently: define

  $\mathcal{S}$ = the free colimit completion of the one-point space $*$

  $\mathcal{S}_* = \mathrm{Hom}_{\mathcal{S}}(*, \mathcal{S})$

  $\rightsquigarrow$ easy to verify $\mathcal{S}_* \xrightarrow{u} \mathcal{S}$ is a left fibration

To define a homotopy coherent functor $\mathrm{Fun}(A,S) \xrightarrow{\int} \mathcal{L}\mathrm{Fib}/A$ use the microcosm principle:

Prop. $\mathrm{coCart}(\mathcal{K}) \xrightarrow{\mathrm{cod}} \mathcal{K}$ is a cartesian fibration of $(\infty,2)$-categories:

- locally a cocartesian fibration of ∞-categories, and whiskering defines a cartesian functor
- globally a cartesian fibration up to homotopy

Cor. Cocartesian cocones lift through $\mathrm{coCart}(\mathcal{K}) \xrightarrow{\mathrm{cod}} \mathcal{K}$

The functor $\mathrm{Fun}(A,S) \xrightarrow{\int} \mathcal{L}\mathrm{Fib}/A$ is defined by one such lift.

The proof that $\mathrm{Fun}(A,S) \xrightarrow{\int} \mathcal{L}\mathrm{Fib}/A$ is fully faithful is a long story that we won't get into here.

Summary:

The 2-category of ∞-categories $\mathcal{CAT}_\infty$ is a weak 2-topos:

(0) a cartesian closed 2-category that has

(1) weak finite limits,

(γ) a duality involution, and

(2) a classifying left fibration.

References:

- Mark Weber "Yoneda structures from 2-toposes"
  Applied Categorical Structures 2007
- Emily Riehl + Dominic Verity "Elements of ∞-category theory"
  draft available at www.math.jhu.edu/~eriehl/elements.pdf

Why might we care that $\mathcal{CAT}_\infty$ is a 2-topos?

For $A \in \mathcal{CAT}_\infty$ define $PA := S^{A^\circ}$.

Declare $A \overset{f}{\to} B \in \mathcal{CAT}_\infty$ to be **admissible** if the module $\mathrm{Hom}_B(f,B)$ is classified by some $B \times A^\circ \to S$.

Note $A \in \mathcal{CAT}_\infty$ is **admissible** just when A^2 is classified by a functor $A \times A^\circ \to S$ which transposes to define $A \xrightarrow{\mathcal{L}} PA$.

Theorem (Weber). Any 2-topos specifies a good Yoneda structure.

Thank you!