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Recent highlights in computer formalized mathematics beyond Lean

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New Perspectives on Stable Homotopy and Beyond

Plan

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   - Rzk proof assistant: Nikolai Kudasov, Violetta Sim, and Benedict Ahrens
   - Formalization in Rzk: Nikolai Kudasov, Emily Riehl, and Jonathan Weinberger
     with contributions from Fredrik Bakke, Tashi Walde, et al
   - Formalization in Agda Flat/Sharp: Samuel Toth with contributions from Fredrik Bakke, Mark Damuni Williams, Rob Schellingerhout

1

The Serre Finiteness Theorem in Cubical Agda

Statement of the Serre Finiteness Theorem

The Serre Finiteness Theorem says that the homotopy groups of spheres are finitely presented abelian groups.

Serre Finiteness Theorem. For all $n, k : \mathbb{N}$,
there merely exists $g, r : \mathbb{N}$ and $\varphi : \mathbb{Z}^r \to \mathbb{Z}^g$ so that $\pi_n S^k \cong \mathbb{Z}^g/\varphi(\mathbb{Z}^r)$.

The (non-unique) homomorphism φ defines a $g \times r$ matrix of integers, which has a unique Smith normal form whose diagonal entries are natural numbers each dividing all subsequent entries:

$$\begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & d_1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & d_t & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$$

$$\pi_n S^k \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_\ell$$

with $1 \neq d_1 \mid \cdots \mid d_\ell$
(possibly including zeros).

Spheres and homotopy groups

The spheres S^n can be defined by iterated suspension starting from $S^{-1} := \emptyset$.

$$\begin{tikzcd} S^{n-1} \arrow[r, "!"] \arrow[d, "!"'] & * \arrow[d] \\ * \arrow[r] & S^n \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

For a pointed homotopy type X, the (underlying set of the) nth homotopy group is the set-truncation (aka the path components) of the homotopy type of pointed maps

$$\pi_n X := \|S^n \to_* X\|_0$$

Stably almost finite implies finitely presentable homotopy

Barton and Campion introduce a predicate on homotopy types called stably almost finite.

A homotopy type is stably almost finite if and only if its homology groups are finitely presented, but the definition of stably almost finite does not reference homology.

Theorem (Barton–Campion). If X is a 1-connected pointed homotopy type and X is stably almost finite, then $\pi_n X$ is finitely presented for all $n : \mathbb{N}$.

The spheres are stably almost finite. (By direct calculation, the homotopy groups of S^0 and S^1 are finitely presentable, while S^k is 1-connected for $k \geq 2$.) Thus, as a corollary:

Serre Finiteness Theorem. For all $n, k : \mathbb{N}$, there merely exists $g, r : \mathbb{N}$ and $\varphi : \mathbb{Z}^r \to \mathbb{Z}^g$ so that $\pi_n S^k \cong \mathbb{Z}^g / \varphi(\mathbb{Z}^r)$.

Base case: $(n-1)$-connected homotopy types

It is easiest to calculate $\pi_n X$ when it is the bottom non-zero homotopy group, so to warm up suppose that X is $(n-1)$-connected and $n \geq 2$.

In the case of a finite CW complex:

Weak Hurewicz Theorem. Let C be an $(n-1)$-connected finite CW complex with $n \geq 2$. Then $\pi_n C$ is finitely presentable.

Proof: There is an n-connected map to C from the cofiber of some $\varphi\colon \vee_R S^n \to \vee_G S^n$ between finite wedges of n-spheres, inducing an isomorphism $\mathbb{Z}[G]/\varphi(\mathbb{Z}[R]) \cong \pi_n C$. $\square$

More generally:

Weak Hurewicz Corollary. Suppose X is an $(n-1)$-connected homotopy type, where $n \geq 2$, with an n-connected map $f\colon C \to X$. Then $\pi_n X$ is finitely presentable.

Proof: $\pi_n C \cong \pi_n X$ and C satisfies the hypotheses of the previous theorem. $\square$

Suspended base case

We have proven:

Weak Hurewicz Corollary. Suppose X is an $(n-1)$-connected homotopy type, where $n \geq 2$, with an n-connected map $f\colon C \to X$. Then $\pi_n X$ is finitely presentable.

It suffices to assume only that some suspension of X has a suitably connected approximation by a finite CW complex:

Theorem B. Suppose X is an $(n-1)$-connected homotopy type, where $n \geq 2$, with an $n+m$-connected map $f\colon C \to \Sigma^m X$ for some m. Then $\pi_n X$ is finitely presentable.

Proof: Since X is $(n-1)$-connected and $n \geq 2$,

$$\pi_n X \cong \pi_{n+m}(\Sigma^m X)$$

by the Freudenthal suspension theorem, and $\Sigma^m X$ is $(n+m-1)$-connected, satisfying the hypotheses of the previous theorem. $\square$

Stably almost finite homotopy types

We have proven:

Theorem B. Suppose X is an $(n-1)$-connected homotopy type, where $n \geq 2$, with an $n+m$-connected map $f\colon C \to \Sigma^m X$ for some m. Then $\pi_n X$ is finitely presentable.

This motivates the definition:

Definition. A homotopy type X is stably almost finite if and only if for all n there merely exists m and a finite CW complex C with an $n+m$-connected map $f\colon C \to \Sigma^m X$.

Thus, we have proven:

Theorem B. Suppose X is an $(n-1)$-connected homotopy type, where $n \geq 2$, that is stably almost finite. Then $\pi_n X$ is finitely presentable.

It remains to reduce the general case to this special case.

Truncations and connected covers

For any pointed homotopy type X and $m : \mathbb{N}$, we may define

$$\begin{tikzcd} X\langle m\rangle \arrow[r, "\epsilon_m"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta_m"] \\ * \arrow[r, "x_0"'] & \|X\|_m \end{tikzcd}$$

where

- $\eta_m : X \to \|X\|_m$ is the m-truncation of X, the universal map to an m-truncated homotopy type $\|X\|_m$, with $\pi_i(\|X\|_m) \cong 0$ for $i > m$.
- Moreover, η_m is m-connected, inducing an isomorphism $\pi_i X \cong \pi_i \|X\|_m$ for $0 \leq i \leq m$ and a surjection on π_{m+1}.
- $\epsilon_m : X\langle m\rangle \to X$ is the m-connected cover of X, the universal pointed map from an m-connected homotopy type $X\langle m\rangle$, with $\pi_i(X\langle m\rangle) \cong 0$ for $1 \leq i < m - 1$.
- Moreover, ϵ_m is $(m-1)$-truncated, inducing an isomorphism $\pi_i(X\langle m\rangle) \cong \pi_i(X)$ for $i > m$.

Proof strategy

For any pointed homotopy type X and $n \geq 2$

$$\pi_n(X\langle n-1\rangle) \cong \pi_n(X)$$

where $X\langle n-1\rangle$ is $(n-1)$-connected.

$$\begin{tikzcd} X\langle n-1\rangle \arrow[r, "\epsilon_{n-1}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta_{n-1}"] \\ * \arrow[r, "x_0"'] & \|X\|_{n-1} \end{tikzcd}$$

Theorem A combines with the already-proven Theorem B to achieve our goal:

Theorem A. If X is stably almost finite and 1-connected, then $X\langle n-1\rangle$ is stably almost finite for any $n \geq 2$.

Theorem B. Suppose X is an $(n-1)$-connected homotopy type, where $n \geq 2$, that is stably almost finite. Then $\pi_n X$ is finitely presentable.

Theorem (Barton–Campion). If X is a 1-connected pointed homotopy type and X is stably almost finite then $\pi_n X$ is finitely presented for all n.

Stability of stably almost finite homotopy types

Finite CW complexes are stably almost finite. The class of stably almost finite homotopy types is closed under:

- suspension
- 2-of-3 for cofiber sequences
- joins, wedge product, smash product, and cartesian product

More elaborate arguments using the Ganea construction show:

- If B is 1-connected and stably almost finite, then ΩB is stably almost finite.
- If A is a finitely presented abelian group and $n \geq 1$, then $K(A, n)$ is stably almost finite (using the previous result and the structure theorem).
- In a fiber sequence $F \to E \to B$, if B is 1-connected and B and F are stably almost finite, E is stably almost finite.
- In a fiber sequence $F \to E \to B$, if E and B are 1-connected and stably almost finite, B is stably almost finite.

Stably almost finite connected covers

Theorem A. If X is stably almost finite and 1-connected, then $X\langle n-1\rangle$ is stably almost finite for any $n \geq 2$.

Proof: We prove by induction on $n \geq 1$, that $X\langle n\rangle$ and $\|X\|_n$ are stably almost finite using the fiber sequences

$$X\langle n\rangle \to X \to \|X\|_n \quad \text{and} \quad K(\pi_n X, n) \to \|X\|_n \to \|X\|_{n-1}.$$

- In the base case, $\|X\|_1$ is contractible and $X\langle 1\rangle \simeq X$.
- So assume $X\langle n-1\rangle$ and $\|X\|_{n-1}$ are stably almost finite.
- By Theorem B, $\pi_n(X\langle n-1\rangle) \cong \pi_n X$ is finitely presentable, so $K(\pi_n X, n)$ is stably almost finite.
- Thus we conclude that $\|X\|_n$ and $X\langle n\rangle$ are stably almost finite from the fiber sequences.

□

Why Cubical Agda?

Why formalize the Serre Finiteness Theorem in Cubical Agda?

Serre Finiteness Theorem. For all $n, k : \mathbb{N}$,
there merely exists $g, r : \mathbb{N}$ and $\varphi : \mathbb{Z}^r \to \mathbb{Z}^g$ so that $\pi_n S^k \cong \mathbb{Z}^g / \varphi(\mathbb{Z}^r)$.

The Serre Finiteness Theorem formalization supplies an element of

$$\Pi_{n,k:\mathbb{N}} \| \Sigma_{g,r:\mathbb{N}} \Sigma_{\varphi:\mathbb{Z}^r \to \mathbb{Z}^g} \pi_n S^k \cong \mathbb{Z}^g / \varphi(\mathbb{Z}^r) \|_{-1},$$

which by (not yet formalized) uniqueness of Smith Normal Forms gives an element of:

$$\Pi_{n,k:\mathbb{N}} \Sigma_{d:\mathsf{List}(\mathbb{N})} \| \pi_n S^k \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_\ell \mathbb{Z} \|_{-1}.$$

Cubical Agda satisfies canonicity (because univalence is constructively provable) meaning that for any particular values of n and k, Cubical Agda can in theory normalize this element to calculate the explicit numerals in the list d determining $\pi_n S^k$.

2

Synthetic ∞-category theory in Rzk and Agda flat/sharp

Semantic setting for synthetic ∞-category theory

Our results about "homotopy types" also hold for objects in any ∞-topos.

In what follows, our results about "synthetic ∞-categories" can be interpreted in simplicial objects in an ∞-topos — which is in particular an ∞-topos!

Thus, we may use the moduli spaces for various propositions from homotopy type theory:

- For any space A, a point in the space

$$\mathsf{isContr}(A) := \Sigma_{x:A}\Pi_{y:A}x = y$$

  provides a proof of the contractibility of A.
- A similar moduli space detects whether a map $f : A \to B$ is an equivalence

$$\mathsf{isEquiv}(f) := \left(\Sigma_{g:B\to A} g \circ f = \mathrm{id}_A\right) \times \left(\Sigma_{h:B\to A} f \circ h = \mathrm{id}_B\right)$$

Simplices from a directed interval

Synthetic ∞-category theory is developed in the context of a directed interval $\mathbb{2}$:

- with distinct endpoints $0, 1 : \mathbb{2}$,
- with a family of propositions $(x \leq y)_{x,y:\mathbb{2}}$, and
- satisfying reflexivity, transitivity, antisymmetry, totality, with min 0 and max 1.

From the interval, (cf. Joyal) we define the standard simplices (and maps between them):

$$\Delta^0 := * \qquad \Delta^2 := \Sigma_{x,y:\mathbb{2}} y \leq x$$

$$\Delta^1 := \mathbb{2} \qquad \Delta^3 := \Sigma_{x,y,z:\mathbb{2}} z \leq y \leq x$$

$$\begin{tikzcd} \Delta^2 \arrow[d, two heads, "(y \leq x)_{x,y:\mathbb{2}}"] & \Delta^3 \arrow[d, two heads, "(z \leq y \leq x)_{x,y,z:\mathbb{2}}"] & \cdots \\ \mathbb{2} \times \mathbb{2} & \mathbb{2} \times \mathbb{2} \times \mathbb{2} & \end{tikzcd}$$

as well as simplicial subspaces such as

$$\Lambda^2_1 := \Sigma_{x,y:\mathbb{2}} (y = 0) \vee (x = 1)$$

$$\partial\Delta^2 := \Sigma_{x,y:\mathbb{2}} (y = 0) \vee (x = y) \vee (x = 1)$$

$$\begin{tikzcd} & \Delta^0 \arrow[rr] \arrow[dl] \arrow[ddr, two heads] \arrow[dr, phantom, "\ulcorner", very near end] & & \Delta^1 \arrow[dl] \arrow[ddl, two heads] & \\ \Delta^1 \arrow[rr] \arrow[drr, two heads] & & \Lambda^2_1 \arrow[rr] \arrow[d, two heads] & & \partial\Delta^2 \arrow[dll, two heads] \\ & & \mathbb{2} \times \mathbb{2} & & \end{tikzcd}$$

Arrows and commutative triangles

Maps out of Δ^1 and Δ^2 define arrows and commutative triangles in a type A.

$$\begin{tikzcd} \mathsf{hom}(x,y) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & A^{\Delta^1} \arrow[d, two heads, "\mathsf{res}"] \\ * \arrow[r, "(x,y)"'] & A \times A \end{tikzcd} \qquad \begin{tikzcd} \mathsf{hom}_2(f,g,h) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & A^{\Delta^2} \arrow[d, two heads, "\mathsf{res}"] \\ * \arrow[r, "(f,g,h)"'] & A^{\partial\Delta^2} \end{tikzcd}$$

```
#def hom (A : U) (x y : A) : U
  := (t : Δ¹) → A [ t ≡ 0₂ ↦ x , t ≡ 1₂ ↦ y ]
```

```
#def hom2 (A : U) (x y z : A)
  (f : hom A x y) (g : hom A y z) (h : hom A x z) : U
  := (( t₁ , t₂) : Δ²) → A
       [ t₂ ≡ 0₂ ↦ f t₁ , t₁ ≡ 1₂ ↦ g t₂ , t₂ ≡ t₁ ↦ h t₂]
```

For example, projection functions define identities and their commutative triangles:

```
#def id-hom (A : U) (x : A) : hom A x x
  := \ _ → x
```

```
#def id-comp-witness (A : U)(x y : A)(f : hom A x y)
  : hom2 A x x y (id-hom A x) f f
  := \ (t , s) → f s
```

Synthetic ∞-categories

A simplicial homotopy type A is a synthetic ∞-category when it satisfies the internal Segal and Rezk completeness conditions.

Definition. A simplicial homotopy type A is an ∞-category if:

- The restriction map res : $A^{\Delta^2} \to A^{\Lambda_1^2}$ is an equivalence.
- The restriction map res : $A \to A^{\square}$ is an equivalence, where $\square$ is the homotopy coherent isomorphism.

We only need the first condition, which equivalently asserts that for any composable pair of arrows $f : \hom_A(x, y)$ and $g : \hom_A(y, z)$, the space of composites is contractible:

```
#def is-segal (A : U) : U
  := (x : A) → (y : A) → (z : A)
  → (f : hom A x y) → (g : hom A y z)
  → is-contr (Σ (h : hom A x z) , hom2 A x y z f g h)
```

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z \end{tikzcd} \quad : A^{\Lambda_1^2}$$

Unique composition in synthetic ∞-categories

In a synthetic ∞-category A, for any composable pair of arrows $f : \hom_A(x, y)$ and $g : \hom_A(y, z)$, the space of composites is contractible:

```
#def is-segal (A : U) : U
  := (x : A) → (y : A) → (z : A)
  → (f : hom A x y) → (g : hom A y z)
  → is-contr (Σ (h : hom A x z) , hom2 A x y z f g h)
```

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ x & & & & z \end{array} \quad : A^{\Lambda^2_1}$$

Thus, A has a well-defined unique composition function constructed from the center of contraction and contracting homotopy of the contractible space of composites:

$$\circ : \hom_A(x, y) \to \hom_A(y, z) \to \hom_A(x, z)$$

```
#def comp-is-segal (A : U) (is-segal-A : is-segal A)
  ( x y z : A) (f : hom A x y) (g : hom A y z)
  : hom A x z
  := first (first (is-segal-A x y z f g))
```

```
#def uniqueness-comp-is-segal (A : U)
  ( is-segal-A : is-segal A) (x y z : A)
  ( f : hom A x y) (g : hom A y z) (h : hom A x z)
  ( alpha : hom2 A x y z f g h)
  : ( comp-is-segal A is-segal-A x y z f g) = h
```

Composition in synthetic ∞-categories is unital and associative

$$\circ : \mathrm{hom}_A(x,y) \to \mathrm{hom}_A(y,z) \to \mathrm{hom}_A(x,z)$$

```
#def comp-is-segal (A : U) (is-segal-A : is-segal A)
  ( x y z : A) (f : hom A x y) (g : hom A y z)
  : hom A x z
  := first (first (is-segal-A x y z f g))
```

```
#def uniqueness-comp-is-segal (A : U)
  ( is-segal-A : is-segal A) (x y z : A)
  ( f : hom A x y) (g : hom A y z) (h : hom A x z)
  ( alpha : hom2 A x y z f g h)
  : ( comp-is-segal A is-segal-A x y z f g) = h
```

Using the unique composition function in a synthetic ∞-category A, it follows that composition is unital and associative:

```
#def id-comp-is-segal (A : U) (is-segal-A : is-segal A)
  ( x y : A) (f : hom A x y)
  : comp-is-segal A is-segal-A x x y (id-hom A x) f = f
  := uniqueness-comp-is-segal A is-segal-A x x y
    ( id-hom A x) f f (id-comp-witness A x y f)
```

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & & y \end{tikzcd} : A^{\Delta^2}$$

Natural transformations

Fix a synthetic ∞-category A and element $a, b : A$. A family of functions

$$\varphi : \Pi_{z:A} \text{hom}_A(z, a) \to \text{hom}_A(z, b)$$

defines the components of a natural transformation between contravariant functors.

Theorem: Any family of functions $\varphi : \prod_{z:A} \text{hom}_A(z, a) \to \text{hom}_A(z, b)$ is natural:

for any $f : \text{hom}_A(x, y)$ and $v : \text{hom}_A(y, a)$

$$f \circ \varphi_y(v) = \varphi_x(f \circ v).$$

Because we have naturality for free, the type of natural transformations is just:

$$\Pi_{z:A} \text{hom}_A(z, a) \to \text{hom}_A(z, b)$$

Statement of the Yoneda lemma

The Yoneda lemma says that evaluation at the identity defines an equivalence between the type of natural transformations $\Pi_{z:A}\mathsf{hom}_A(z,a) \to \mathsf{hom}_A(z,b)$ and the type of arrows $\mathsf{hom}_A(a,b)$.

```
#def evid (A : U) (a b : A)
  : ( ( z : A) → hom A z a → hom A z b) → hom A a b
  := \ φ → φ a (id-hom A a)
```

This allows us to state the Yoneda lemma:

```
#def yoneda-lemma uses (funext)
  ( A : U) (is-segal-A : is-segal A) (a b : A)
  : is-equiv ((z : A) → hom A z a → hom A z b) (hom A a b)
    ( evid A a b)
```

The inverse Yoneda map

To prove that evid : $(\Pi_{z:A}\mathsf{hom}_A(z,a) \to \mathsf{hom}_A(z,b)) \to \mathsf{hom}_A(a,b)$ is an equivalence, we first define the inverse map:

$$\mathsf{yon} : \mathsf{hom}_A(a,b) \to (\Pi_{z:A}\mathsf{hom}_A(z,a) \to \mathsf{hom}_A(z,b))$$

Given $v : \mathsf{hom}_A(a,b)$, we define a natural transformation

$$\mathsf{yon}\ v : \Pi_{z:A}\mathsf{hom}_A(z,a) \to \mathsf{hom}_A(z,b)$$

by postcomposing with v.

```
#def yon (A : U) (is-segal-A : is-segal A) (a b : A)
  : hom A a b → ((z : A) → hom A z a → hom A z b)
  := \ v z f → comp-is-segal A is-segal-A z a b f v
```

The proof of the Yoneda lemma

To prove that the functions

$$(\Pi_{z:A}\mathsf{hom}_A(z,a) \to \mathsf{hom}_A(z,b)) \underset{\mathsf{yon}}{\overset{\mathsf{evid}}{\rightleftarrows}} \mathsf{hom}_A(a,b)$$

are inverse equivalences, we consider their composites.

Given $v : \mathsf{hom}_A(a,b)$, by definition $\mathsf{evid}(\mathsf{yon}\ v)$ reduces to $\mathsf{id}_a \circ v$. Thus, $\mathsf{evid}(\mathsf{yon}\ v) : \mathsf{hom}_A(a,b)$ equals v by unitality of composition: $\mathsf{evid}(\mathsf{yon}\ v) = v$.

This proves that yon is right inverse to evid. It remains only to prove that yon is also left inverse.

```
#def yoneda-lemma uses (funext)
  ( A : U) (is-segal-A : is-segal A) (a b : A)
  : is-equiv ((z : A) → hom A z a → hom A z b) (hom A a b)
    ( evid A a b)
  :=
    ( ( yon A is-segal-A a b
      , ?)
    , ( yon A is-segal-A a b
      , id-comp-is-segal A is-segal-A a b))
```

The proof of the Yoneda lemma

The proof that yon is left inverse to evid amounts to showing for $x : A$ and $f : \text{hom}_A(x, a)$ that $\text{yon}(\text{evid}\ \varphi)_x(f)$ equals $\varphi_x(f) : \text{hom}_A(x, b)$.

```
#def yon-evid-twice-pointwise (A : U)
  ( is-segal-A : is-segal A) (a b : A)
  ( φ : (z : A) → hom A z a → hom A z b)
  ( x : A) (f : hom A x a)
  : ( ( yon A is-segal-A a b) ((evid A a b) φ)) x f = φ x f
  :=
  concat (hom A x b)
  ( ( ( yon A is-segal-A a b) ((evid A a b) φ)) x f)
  ( φ x (comp-is-segal A is-segal-A x a a f (id-hom A a)))
  ( φ x f)
  ( naturality-contravariant-fiberwise-transformation
    A is-segal-A a b x a f (id-hom A a) φ)
  ( ap (hom A x a) (hom A x b)
    ( comp-is-segal A is-segal-A x a a f (id-hom A a))
    ( f) (φ x)
    ( comp-id-is-segal A is-segal-A x a f))
```

By definition $\text{yon}(\text{evid}\ \varphi)_x(f)$ reduces to $f \circ \varphi_a(\text{id}_a)$. By naturality of φ, this equals $\varphi_x(f \circ \text{id}_a)$, which then equals $\varphi_x(f)$ by the other identity law:

$$\text{yon}(\text{evid}\ \varphi)_x(f) = f.$$

By function extensionality, this completes the proof of the Yoneda lemma.

Agda synthetic categories library

The proof assistant RZK implements the original (Riehl–Shulman) simplicial type theory, not its later extensions by Weaver–Licata or Gratzer–Weinberger–Buchholtz.

tl;dr: Proofs in RZK are simple, conceptual, and short — but not all theorems about ∞-categories can be correctly stated (or proven).

A more comprehensive synthetic theory of ∞-categories requires modalities, like the discrete $\dashv$ codiscrete modalities $\flat \dashv \sharp$ of AGDA FLAT/SHARP.

The Synthetic (∞-)Categories library in AGDA FLAT/SHARP was initiated a few months ago by Samuel Toth and is expanding very rapidly...

Strange new universes of synthetic mathematics

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WILL MACHINES CHANGE MATHEMATICS?

MAIA FRASER, ANDREW GRANVILLE, MICHAEL H. HARRIS, COLIN MCLARTY,
EMILY RIEHL, AND AKSHAY VENKATESH
(Guest Editorial Committee)

STRANGE NEW UNIVERSES: PROOF ASSISTANTS AND SYNTHETIC FOUNDATIONS

MICHAEL SHULMAN

Abstract. Existing computer programs called proof assistants can verify the correctness of mathematical proofs but their specialized proof languages present a barrier to entry for many mathematicians. Large language models have the potential to lower this barrier, enabling mathematicians to interact with proof assistants in a more familiar vernacular. Among other advantages, this may allow mathematicians to explore radically new kinds of mathematics using an LLM-powered proof assistant to train their intuitions as well as ensure their arguments are correct. Existing proof assistants have already played this role for fields such as homotopy type theory.

"Out of nothing I have created a strange new universe."
– János Bolyai, one of the inventors of non-Euclidean geometry

"I believe that future mathematics software will combine the ease of interaction of an LLM with the near-absolute trustworthiness of a proof assistant.

...just as a present-day mathematician can exploit a computer's calculational ability to explore numerical realms undreamed-of decades ago, these imagined future mathematicians may exploit a computer's logical ability to explore conceptual realms undreamed-of today.

...What new kinds of mathematics, then, may be waiting for us to explore with the proof assistants of tomorrow?"

— Michael Shulman

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Synthetic ∞-category theory:

- Survey: Synthetic Perspectives on Spaces and Categories, ICM 2026
- Paper: Formalizing the ∞-Categorical Yoneda Lemma, CPP 2024
- Games: Rzk Warmup Game and Yoneda Game
- Formalization: Simplicial HoTT and synthetic ∞-categories library in Rzk
- Formalization: Synthetic categories library in Agda

Thank you!