Emily Riehl

Johns Hopkins University

On the art of giving the same name to different things

SUMS 2023: Math and Language

...la Mathématique est l'art de donner le même nom à des choses différentes.

...mathematics is the art of giving the same name to different things.

— Henri Poincaré
"L'avenir des mathématiques"
Science et Méthode
Flammarion, Paris, 1908.

Plan

Equality

$=$

Isomorphism

$\cong$

Equivalence

$\simeq$

Identification

$=$

1

Equality

The traditional view of equality

Reflexivity:
anything is equal to itself.

$$\forall x,\ x = x$$

Indiscernibility of Identicals:
if two things are equal, then they have exactly the same properties.

$$\forall x, y,\ (x = y) \to (\forall P, P(x) \leftrightarrow P(y))$$

Symmetry and Transitivity

Using

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Different things that deserve the same name

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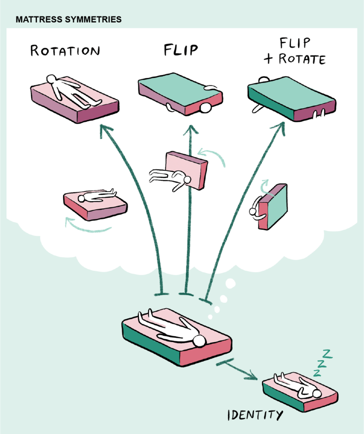
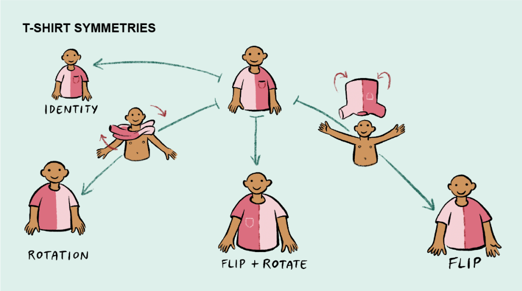


images by Matteo Farinella



2

Isomorphism

$\cong$

Isomorphic = same + shape

Some different things deserve the same name because they have the "same shape."

ίσος "equal" + μορφή "shape"

We seek a unifying language to describe what it means for things to have the "same shape" no matter what kind of objects they are.

Category

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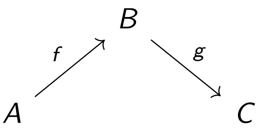

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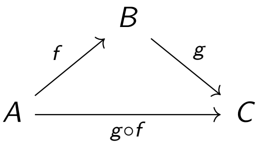

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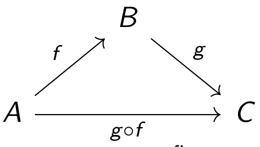

- and each object has an reflexivity arrow $A \xrightarrow{\mathrm{refl}_A} A$

for which the composition operation is associative and unital.

Isomorphism in a category

A category has

- objects: $A, B, C \ldots$ and
- arrows: $A \xrightarrow{f} B$, $B \xrightarrow{g} C$.

Objects A and B in a category are isomorphic

if there exist arrows $f : A \to B$ and $g : B \to A$

so that $g \circ f = \mathrm{refl}_A$ and $f \circ g = \mathrm{refl}_B$.

$$A \cong B$$

Categorifying arithmetic

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$$\| $$

$$(A \times B) + (A \times C)$$

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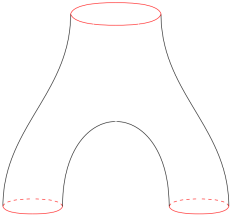

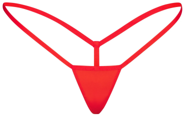

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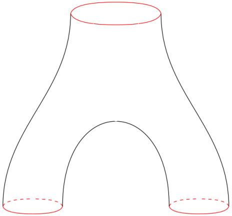
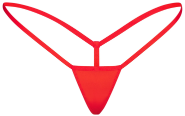
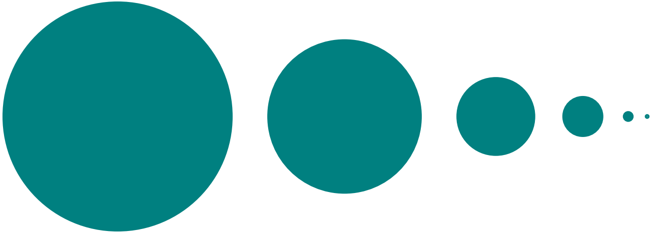

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— and very redundant.

The category of natural numbers and their symmetries contains the same information, much more efficiently packaged.

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There are two standard approaches to linear algebra:

- using matrices of arbitrary dimension
- using linear transformations between vector spaces

and the general theory can be developed from either perspective.

3

Equivalence

$\simeq$

Equivalence = equal + worth

A 2-category has

- objects: $A, B, C \ldots$
- 1-arrows: $A \xrightarrow{f} B$, $B \xrightarrow{h} C$ and
- 2-arrows: $A \underset{k}{\overset{f}{\Downarrow \alpha}} B$

Objects A and B in a 2-category are equivalent

if there exist 1-arrows $f\colon A \to B$ and $g\colon B \to A$

and 2-arrows $A \underset{\mathrm{refl}_A}{\overset{g \circ f}{\Downarrow \alpha}} A$ and $B \underset{\mathrm{refl}_B}{\overset{f \circ g}{\Downarrow \beta}} B$

so that $\alpha\colon g \circ f \cong \mathrm{refl}_A$ and $\beta\colon f \circ g \cong \mathrm{refl}_B$.

$$A \simeq B$$

A contracting homotopy equivalence

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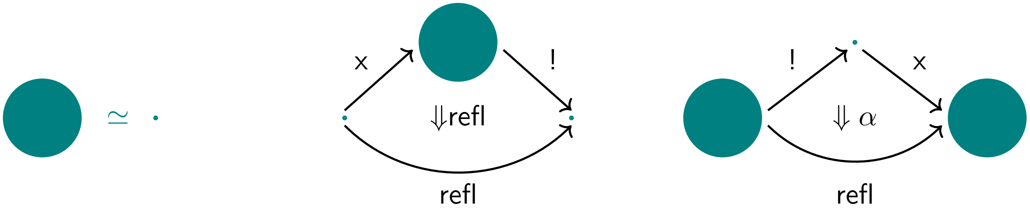

Problems

- This doesn't stop here! The best notion of sameness for 2-categories isn't equivalence in the sense just defined but in a weaker sense that requires a 3-category. But then 3-categories are equivalent in a sense defined using a 4-category, and so on ...

- Higher category theory no longer provides a single meaning of when one thing is the same as another thing but rather a hierarchy of different meanings depending on how complex the objects are, as governed by what sort of categories they belong to.

- Most seriously, indiscernibility of identicals fails for objects that are isomorphic or equivalent but not equal!

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Q: Is 3 an element of 17?

For the von Neumann naturals yes, but for the Zermelo naturals no!
— Paul Benacerraf "What numbers could not be"

4

Identification

Identity Types

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A term $x : A$ in a type then provides a proof of the encoded statement.

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Identity types are governed by the following rules:

- For any type A and terms $x, y : A$, there is a type $x =_A y$.
- For any type A and term $x : A$, there is a term $\mathrm{refl}_x : x =_A x$.
- For any type $P(x, y, p)$ defined using terms $x, y : A$ and $p : x =_A y$,
  - if there is a term $d(x) : P(x, x, \mathrm{refl}_x)$ for all $x : A$,
  - then there is a term $J_d(x, y, p) : P(x, y, p)$ for all $x, y : A$, $p : x =_A y$.

No nonsense: it's only meaningful to identify things in a common type.

Reflexivity: anything is identifiable with itself.

Indiscernibility of Identicals: if two things are equal,
then they have exactly the same properties.

Univalence

The univalence axiom relates the identity types in the universe of all types $\mathcal{U}$ to equivalences between types.

"Identity is equivalent to equivalence."

$$\text{univalence} : (A =_{\mathcal{U}} B) \simeq (A \simeq_{\mathcal{U}} B)$$

"When I decided to check something in the Russian translation of the Boardman and Vogt book *Homotopy Invariant Algebraic Structures on Topological Spaces* I discovered that in this book the term 'faithful functor' was translated as 'univalent functor.'

унивалентный функтор

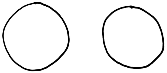
Since I have tried to read this book in my youth many times there was probably another meaning associated in my mind with the word 'univalent' — 'faithful'.

Indeed these foundations seem to be faithful to the way in which I think about mathematical objects in my head."

— Vladimir Voevodsky, "Univalent Foundations — new type-theoretic foundations of mathematics," Talk at IHP, Paris on April 22, 2014

Consequences of Univalence

The things that deserve the same name:

- 
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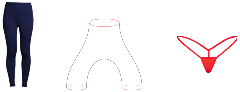
- 
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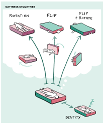
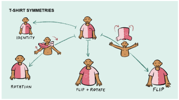
- 



- the categories of finite sets and of natural numbers
- abstract and concrete linear algebra

are terms belonging to a common type.

As a consequence of the univalence axiom:

identifications — that is, proofs of identity —

recover exactly the notions of sameness previously introduced.

Hierarchies of complexity of identifications

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*Unique up to homotopy: a contractible type has a term and all terms are identifiable.

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- $\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$
- A type is an n-type if its identity types are $n-1$-types.

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By univalence: $\mathbb{N}$ is a set, so $2 \times (3+4) = (2 \times 3) + (2 \times 4)$ is a proposition.
Group is a 1-type, so $K_4 = K_4$ is a set.
1-Cat is a 2-type, so Vect $=$ Mat is a 1-type.

Conclusions

Equality $\rightsquigarrow$ Isomorphism $\rightsquigarrow$ Equivalence $\rightsquigarrow$ Identification

- While the traditional notion of equality is too narrow, its defining principles are worth preserving.
- While the categorical notions of isomorphism and equivalence identify objects that have the "same shape" or have "equal worth," they require increasingly higher-dimensional data as the objects become more complex.
- The type theoretic concept of identification is specified by rules that demand:
  - no nonsense: it's only meaningful to identify things of the same type,
  - reflexivity: everything is identified with itself, and
  - indiscernibility of identicals: if two things are identifiable, they have exactly the same properties.
- In the presence of the univalence axiom, identifications specialize to the "correct" notions of sameness for objects of each type.

Thank you!