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On the art of giving the same name to different things

Calgary Math and Philosophy Lectures

...la Mathématique est l'art de donner le même nom à des choses différentes.

...mathematics is the art of giving the same name to different things.

— Henri Poincaré
"L'avenir des mathématiques"
Science et Méthode
Flammarion, Paris, 1908.

Plan

Equality
$=$

Isomorphism
$\cong$

Equivalence
$\simeq$

Identification
$=$

1

Equality

The traditional view of equality

Reflexivity:
anything is equal to itself.

$$\forall x,\ x = x$$

Indiscernibility of Identicals:
if two things are equal, then they have exactly the same properties.

$$\forall x, y,\ (x = y) \to (\forall P, P(x) \leftrightarrow P(y))$$

Symmetry and Transitivity

Using

- reflexivity: anything is equal to itself; and
- indiscernibility of identicals: if two things are equal, then they have exactly the same properties.

one can deduce:

Symmetry: if $x = y$ then $y = x$.

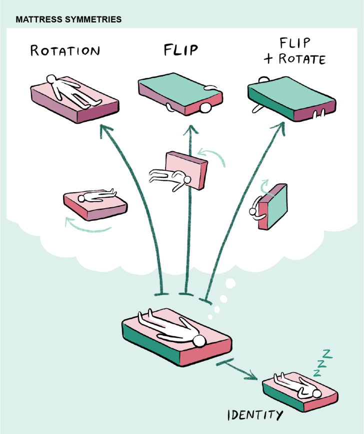
Proof: Assume $x = y$. Then x and y must have exactly the same properties. In particular, since $x = x$ we must also have $y = x$. □

Transitivity: if $x = y$ and $y = z$ then $x = z$.

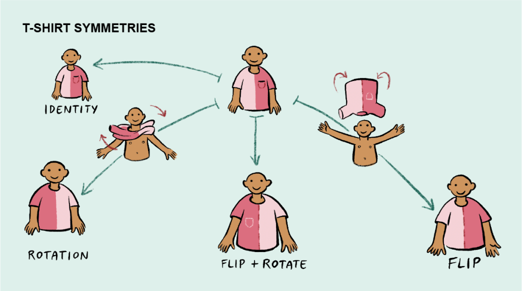
Proof: Assume $x = y$. Then x and y must have exactly the same properties. In particular, if $y = z$ then $x = z$. □

Different things that deserve the same name

Different things that deserve the same name



images by Matteo Farinella



2

Isomorphism

$\cong$

Isomorphic = same + shape

Some different things deserve the same name because they have the "same shape."

$\acute{\iota}\sigma$ος "equal" + $\mu o \rho \phi \acute{\eta}$ "shape"

We seek a unifying language to describe what it means for things to have the "same shape" no matter what kind of objects they are.

Category

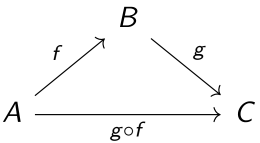
> A category frames a possible template for a mathematical theory: the theory should have nouns, the mathematical objects, and verbs, the transformations between them, depicted as arrows. — Barry Mazur

A category has

- objects: $A, B, C \dots$ and
- arrows: $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, each with a specified source and target

so that

- each pair of composable arrows has a composite arrow



- and each object has an reflexivity arrow $A \xrightarrow{\mathrm{refl}_A} A$

for which the composition operation is associative and unital.

Isomorphism in a category

A category has

- objects: $A, B, C \dots$ and
- arrows: $A \xrightarrow{f} B$, $B \xrightarrow{g} C$.

Objects A and B in a category are isomorphic

if there exist arrows $f : A \to B$ and $g : B \to A$

so that $g \circ f = \mathrm{refl}_A$ and $f \circ g = \mathrm{refl}_B$.

$$A \cong B$$

Categorifying arithmetic

Why is $2 \times (3+4) = (2 \times 3) + (2 \times 4)$?

What even are 2, 3, and 4 ?

$$A = \{ \ast \quad \star \}, \qquad B = \left\{ \begin{matrix} \sharp \\ \flat \\ \natural \end{matrix} \right\}, \qquad C = \left\{ \begin{matrix} \spadesuit & \heartsuit \\ \diamondsuit & \clubsuit \end{matrix} \right\}$$

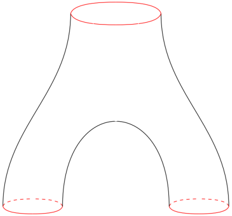
$$B + C = \left\{ \begin{matrix} \sharp \quad \flat \quad \spadesuit \quad \heartsuit \\ \natural \quad \diamondsuit \quad \clubsuit \end{matrix} \right\}, \qquad A \times B = \left\{ \begin{matrix} (\ast, \sharp) & (\star, \sharp) \\ (\ast, \flat) & (\star, \flat) \\ (\ast, \natural) & (\star, \natural) \end{matrix} \right\}$$

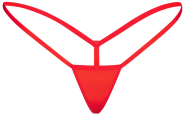
$$A \times (B + C) = \left\{ \begin{matrix} (\ast, \sharp) & (\star, \sharp) \\ (\ast, \flat) & (\star, \flat) \\ (\ast, \natural) & (\star, \natural) \\ (\ast, \spadesuit) & (\star, \spadesuit) \\ (\ast, \heartsuit) & (\star, \heartsuit) \\ (\ast, \diamondsuit) & (\star, \diamondsuit) \\ (\ast, \clubsuit) & (\star, \clubsuit) \end{matrix} \right\} \cong \left\{ \begin{matrix} (\ast, \sharp) \quad (\ast, \flat) \quad (\ast, \spadesuit) \quad (\ast, \heartsuit) \\ (\ast, \natural) \quad (\ast, \diamondsuit) \quad (\ast, \clubsuit) \\ (\star, \sharp) \quad (\star, \flat) \quad (\star, \spadesuit) \quad (\star, \heartsuit) \\ (\star, \natural) \quad (\star, \diamondsuit) \quad (\star, \clubsuit) \end{matrix} \right\}$$

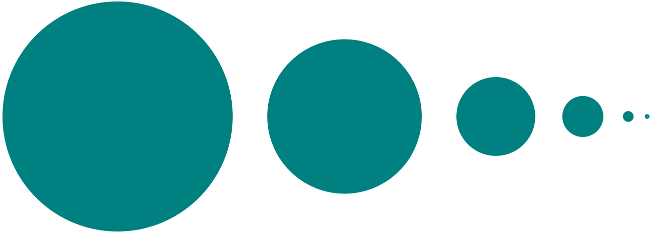
$$\| $$

$$(A \times B) + (A \times C)$$

Different things that deserve the same name







Different things that deserve the same name

The category of finite sets and isomorphisms is indescribably large

— and very redundant.

The category of natural numbers and their symmetries contains the same information, much more efficiently packaged.

There are two standard approaches to linear algebra:

- using matrices of arbitrary dimension
- using linear transformations between vector spaces

and the general theory can be developed from either perspective.

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Equivalence

$\simeq$

Equivalence = equal + worth

A 2-category has

- objects: $A, B, C \dots$
- 1-arrows: $A \xrightarrow{f} B$, $B \xrightarrow{h} C$ and
- 2-arrows: $A \underset{k}{\overset{f}{\Downarrow \alpha}} B$

Objects A and B in a 2-category are equivalent

if there exist 1-arrows $f\colon A \to B$ and $g\colon B \to A$

and 2-arrows $A \underset{\mathrm{refl}_A}{\overset{g \circ f}{\Downarrow \alpha}} A$ and $B \underset{\mathrm{refl}_B}{\overset{f \circ g}{\Downarrow \beta}} B$

so that $\alpha\colon g \circ f \cong \mathrm{refl}_A$ and $\beta\colon f \circ g \cong \mathrm{refl}_B$.

$$A \simeq B$$

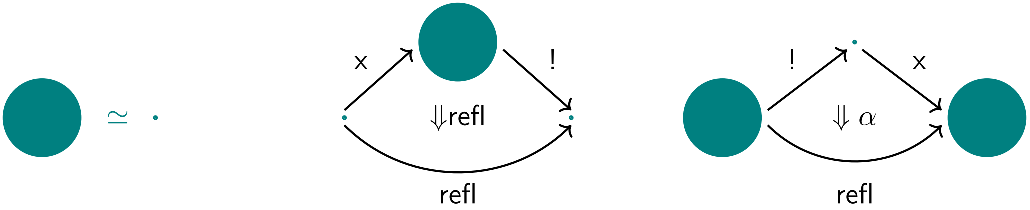
A contracting homotopy equivalence

Objects A and B in a 2-category are equivalent

if there exist 1-arrows $f\colon A \to B$ and $g\colon B \to A$

and 2-arrows $A \underset{\mathrm{refl}_A}{\overset{g \circ f}{\Downarrow \alpha}} A$ and $B \underset{\mathrm{refl}_B}{\overset{f \circ g}{\Downarrow \beta}} B$

so that $\alpha\colon g \circ f \cong \mathrm{refl}_A$ and $\beta\colon f \circ g \cong \mathrm{refl}_B$.



Problems

- This doesn't stop here! The best notion of sameness for 2-categories isn't equivalence in the sense just defined but in a weaker sense that requires a 3-category. But then 3-categories are equivalent in a sense defined using a 4-category, and so on ...
- Higher category theory no longer provides a single meaning of when one thing is the same as another thing but rather a hierarchy of different meanings depending on how complex the objects are, as governed by what sort of categories they belong to.
- Most seriously, indiscernibility of identicals fails for objects that are isomorphic or equivalent but not equal!

Q: Is 3 an element of 17?

For the von Neumann naturals yes, but for the Zermelo naturals no!
— Paul Benacerraf "What numbers could not be"

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Identification

Identity Types

In type theory mathematical sentences take the form of types A, B, C.
A term $x : A$ in a type then provides a proof of the encoded statement.

Identity types are governed by the following rules:

- For any type A and terms $x, y : A$, there is a type $x =_A y$.
- For any type A and term $x : A$, there is a term $\mathrm{refl}_x : x =_A x$.
- For any type $P(x, y, p)$ defined using terms $x, y : A$ and $p : x =_A y$,
    - if there is a term $d(x) : P(x, x, \mathrm{refl}_x)$ for all $x : A$,
    - then there is a term $J_d(x, y, p) : P(x, y, p)$ for all $x, y : A$, $p : x =_A y$.

No nonsense: it's only meaningful to identify things of the same type.

Reflexivity: anything is identifiable with itself.

Indiscernibility of Identicals: if two things are equal, then they have exactly the same properties.

Univalence

The univalence axiom relates the identity types in the universe of all types $\mathcal{U}$ to equivalences between types.

"Identity is equivalent to equivalence."

$$\text{univalence} : (A =_{\mathcal{U}} B) \simeq (A \simeq_{\mathcal{U}} B)$$

"When I decided to check something in the Russian translation of the Boardman and Vogt book *Homotopy Invariant Algebraic Structures on Topological Spaces* I discovered that in this book the term 'faithful functor' was translated as 'univalent functor.'

унивалентный функтор

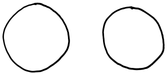
Since I have tried to read this book in my youth many times there was probably another meaning associated in my mind with the word 'univalent' — 'faithful'.

Indeed these foundations seem to be faithful to the way in which I think about mathematical objects in my head."

— Vladimir Voevodsky, "Univalent Foundations — new type-theoretic foundations of mathematics," Talk at IHP, Paris on April 22, 2014

Consequences of Univalence

The things that deserve the same name:

- 
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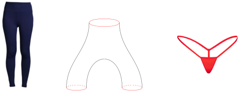
- 
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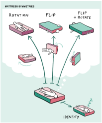
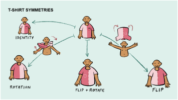
- $2 \times (3 + 4)$ and $(2 \times 3) + (2 \times 4)$
- 



- the categories of finite sets and of natural numbers
- abstract and concrete linear algebra

are terms belonging to a common type.

As a consequence of the univalence axiom:

identifications — that is, proofs of identity — recover exactly the notions of sameness previously introduced.

Hierarchies of complexity of identifications

As a consequence of the univalence axiom:

identifications — that is, proofs of identity —
recover exactly the notions of sameness previously introduced.

- A type is contractible if it has a unique* term.
- A type is a proposition if its identity types are contractible.
- A type is a set if its identity types are propositions.
- ⋮ ⋮
- A type is an n-type if its identity types are $n-1$-types.

*Unique up to homotopy: a contractible type has a term and all terms are identifiable.

By univalence: $\mathbb{N}$ is a set, so $2 \times (3+4) = (2 \times 3) + (2 \times 4)$ is a proposition.
Group is a 1-type, so $K_4 = K_4$ is a set.
1-Cat is a 2-type, so Vect $=$ Mat is a 1-type.

Conclusions

Equality $\rightsquigarrow$ Isomorphism $\rightsquigarrow$ Equivalence $\rightsquigarrow$ Identification

- While the traditional notion of equality is too narrow, its defining principles are worth preserving.
- While the categorical notions of isomorphism and equivalence identify objects that have the "same shape" or have "equal worth," they require increasingly higher-dimensional data as the objects become more complex.
- The type theoretic concept of identification is specified by rules that demand:
  - no nonsense: it's only meaningful to identify things of the same type,
  - reflexivity: everything is identified with itself, and
  - indiscernibility of identicals: if two things are identifiable, they have exactly the same properties.
- In the presence of the univalence axiom, identifications specialize to the "correct" notions of sameness for objects of each type.

Thank you!