Emily Riehl

Johns Hopkins University

Formalizing post-rigorous mathematics

Workshop: bridging between informal and formal

Pre-rigorous, rigorous, and post-rigorous mathematics

The phrase post-rigorous mathematics refers to Terry Tao's blog post "There's more to mathematics than rigour and proofs":

> *One can roughly divide mathematical education into three stages:*
>
> 1. *The "pre-rigorous" stage, in which mathematics is taught in an informal, intuitive manner, based on examples, fuzzy notions, and hand-waving. ...The emphasis is more on computation than on theory. This stage generally lasts until the early undergraduate years.*
> 2. *The "rigorous" stage, in which one is now taught that in order to do maths "properly", one needs to work and think in a much more precise and formal manner ...The emphasis is now primarily on theory; and one is expected to be able to comfortably manipulate abstract mathematical objects without focusing too much on what such objects actually "mean". This stage usually occupies the later undergraduate and early graduate years.*
> 3. *The "post-rigorous"stage, in which one has grown comfortable with all the rigorous foundations of one's chosen field, and is now ready to revisit and refine one's pre-rigorous intuition on the subject, but this time with the intuition solidly buttressed by rigorous theory. ... The emphasis is now on applications, intuition, and the "big picture". This stage usually occupies the late graduate years and beyond.*

Post-rigorous mathematics

The point of rigour is not to destroy all intuition; instead, it should be used to destroy bad intuition while clarifying and elevating good intuition. It is only with a combination of both rigorous formalism and good intuition that one can tackle complex mathematical problems; one needs the former to correctly deal with the fine details, and the latter to correctly deal with the big picture. Without one or the other, you will spend a lot of time blundering around in the dark (which can be instructive, but is highly inefficient). So once you are fully comfortable with rigorous mathematical thinking, you should revisit your intuitions on the subject and use your new thinking skills to test and refine these intuitions rather than discard them. ...

The ideal state to reach is when every heuristic argument naturally suggests its rigorous counterpart, and vice versa. Then you will be able to tackle maths problems by using both halves of your brain at once — i.e., the same way you already tackle problems in "real life".

— *Terry Tao*

Post-rigorous mathematics in practice

Unfortunately, mathematicians often fail to operate in Tao's post-rigorous ideal state. In practice, a proof might be called "post-rigorous" if:

- The argument is not explained in full detail.
- Some claims made as part of the argument may not quite be true as stated.
- Nevertheless, the proof is "morally correct."

Tao continuous:

> *It is perhaps worth noting that mathematicians at all three of the above stages of mathematical development can still make formal mistakes in their mathematical writing. However, the nature of these mistakes tends to be rather different, depending on what stage one is at ... [and] can lead to the phenomenon (which can often be quite puzzling to readers at earlier stages of mathematical development) of a mathematical argument by a post-rigorous mathematician which locally contains a number of typos and other formal errors, but is globally quite sound, with the local errors propagating for a while before being cancelled out by other local errors.*

Plan

1. Literature survey

2. A case study

1

Literature survey

An attempted proof of Grothendieck's homotopy hypothesis

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES
VOL. XXXII-1 (1991)

∞-GROUPOIDS AND HOMOTOPY TYPES

by M.M. KAPRANOV and V.A. VOEVODSKY

RÉSUMÉ. Nous présentons une description de la categorie homotopique des CW-complexes en termes des ∞-groupoïdes. La possibilité d'une telle description a été suggérée par A. Grothendieck dans son memoire "A la poursuite des champs".

It is well-known [GZ] that CW-complexes X such that $\pi_i(X,x) = 0$ for all $i \geq 2$, $x \in X$, are described, at the homotopy level, by groupoids. A. Grothendieck suggested, in his unpublished memoir [Gr], that this connection should have a higher-dimensional generalisation involving polycategories. viz. polycategorical analogues of groupoids. It is the purpose of this paper to establish such a generalisation.

- 15 statements =
  4 theorems
  \+ 9 propositions
  \+ 1 lemma
  \+ 1 corollary
- 5 short "obvious" proofs + 3 proofs

- Carlos Simpson's "Homotopy types of strict 3-groupoids" (1998) shows that the 3-type of S^2 can't be realized by a strict 3-groupoid — contradicting the last corollary
- But no explicit mistake was found. Voevodsky: "I was sure that we were right until the fall of 2013 (!!)"

A sociological problem

MATHEMATICS

The Origins and Motivations of Univalent Foundations

A Personal Mission to Develop Computer Proof Verification to Avoid Mathematical Mistakes

By Vladimir Voevodsky • Published 2014

"A technical argument by a trusted author, which is hard to check and looks similar to arguments known to be correct, is hardly ever checked in detail."

Avoiding a precise definition of ∞-categories

The precursor to Jacob Lurie's *Higher Topos Theory* is a 2003 preprint On ∞-Topoi, which avoids using a precise definition of $(\infty, 1)$-categories aka ∞-categories[1]:

> *We will begin in §1 with an informal review of the theory of ∞-categories. There are many approaches to the foundation of this subject, each having its own particular merits and demerits. Rather than single out one of those foundations here, we shall attempt to explain the ideas involved and how to work with them. The hope is that this will render this paper readable to a wider audience, while experts will be able to fill in the details missing from our exposition in whatever framework they happen to prefer.*

Perlocutions of this form are quite common in the field — however the book *Higher Topos Theory* does not proceed in this manner, instead proving theorems for a concrete model of ∞-categories.

[1] In the parlance of the field, selecting a set-theoretic definition of ∞-categories is referred to as "choosing a model."

A proof(?) of the cobordism hypothesis

The cobordism hypothesis classifies (fully-extended) topological quantum field theories, which are functors indexed by a suitably-defined higher category of cobordisms between framed n-manifolds with corners. In a celebrated expository article on the subject, Dan Freed writes:

> *The cobordism hypothesis was conjectured by Baez-Dolan in the mid 1990s. It has now been proved by Hopkins-Lurie in dimension two and by Lurie in higher dimensions. There are many complicated foundational issues which lie behind the definitions and the proof, and only a detailed sketch has appeared so far.*[1]

The footnote elaborates:

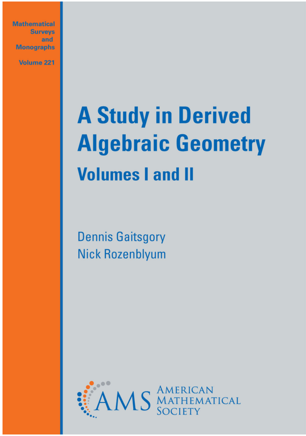
> [1] *Nonetheless, we use "theorem" and its synonyms in this manuscript. The foundations are rapidly being filled in and alternative proofs have also been carried out, though none has yet appeared in print.*

There seems to be no clear consensus on this point of view: a mathOVERFLOW question "What is the status of the cobordism hypothesis?" asked a bit over a year ago remains open.

A conjectural(?) study in derived algebraic geometry

A two-volume study in derived algebraic geometry runs to nearly 1000 pages. Much of the first volume is devoted to developing necessary preliminary results in $(\infty, 1)$-category theory and $(\infty, 2)$-category theory, and includes the following disclaimer:

> *Unfortunately, the existing literature on $(\infty, 2)$-categories does not contain the proofs of all the statements that we need. We decided to leave some of the statements unproved, and supply the corresponding proofs elsewhere (including the proofs here would have altered the order of the exposition, and would have come at the expense of clarity).*



This is followed by a list of seven unproved statements.

Obstructions to formalization?

How might this literature read differently in a future where mathematicians are expected to work interactively with a computer proof assistant?

- An incorrect proof should not be formalizable — which is of course a good thing. And perhaps the process of formalization would help identify the error by calling attention to a subtle obstacle to be overcome.
- If it is undesirable to give a precise construction of a mathematical notion (eg of the $(\infty, 2)$-category of ∞-categories), one could instead axiomatize the necessary properties (and hope that the theory is not vacuous).
- Sketch proofs will be harder to implement, as a proof assistant will require clearer definitions and scaffolding. But a formalized sketch, will make it much clearer what gaps remain in the proof.
- A proof modulo unproven conjectures should be formalizable, provided those conjectures and clearly stated in exactly the way they are used.

2

A case study

Definitions

An ∞-category A is pointed if it has an object $*$ that is both initial and terminal: for all objects a, there exist unique morphisms $* \to a$ and $a \to *$.

A pointed ∞-category A is stable if

- every morphism $f : x \to y$ has both a fiber and a cofiber:

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "f"] \\ * \arrow[r] & y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \arrow[d] \\ * \arrow[r] & \mathrm{cofib}(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

- and fiber and cofiber squares coincide.

A proof?

A pointed ∞-category A is stable if every morphism $f : x \to y$ has both a fiber and a cofiber

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* \arrow[r] & y
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "f"] \arrow[d] & y \arrow[d] \\
* \arrow[r] & \mathrm{cofib}(f) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

and fiber and cofiber squares coincide.

4.4.8. Proposition (pullbacks and pushouts in stable ∞-categories). *A stable ∞-category admits all pushouts and all pullbacks, and moreover, a square is pushout if and only if it is a pullback.*

Proof. Given a generic family of cospans $g \vee f : X \to A^{\lrcorner}$ in A, form the cofiber of f followed by the fiber of the composite map $qg : c \to a \to \mathrm{cofib}\, f$:

$$\begin{tikzcd}
\mathrm{fib}(qg) \arrow[r, dashed, "u"] \arrow[d, dashed, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, dashed] \\
c \arrow[r, "g"'] & a \arrow[r, dashed, "q"'] & \mathrm{cofib}(f) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{4.4.9}$$

By Definition 4.4.6(ii), the cofiber sequence $b \to a \to \mathrm{cofib}(f)$ is also a fiber sequence. By the pullback cancelation result of Proposition 4.3.11, we conclude that $\mathrm{fib}(qg)$ computes the pullback of the cospan $g \vee f$.

An attempt at rigor

4.4.8. Proposition (pullbacks and pushouts in stable ∞-categories). *A stable ∞-category admits all pushouts and all pullbacks, and moreover, a square is pushout if and only if it is a pullback.*

Proof. Given a generic family of cospans $g \vee f \colon X \to A^{\lrcorner}$ in A, form the cofiber of f followed by the fiber of the composite map $qg \colon c \to a \to \operatorname{cofib} f$:

$$\begin{tikzcd}
\operatorname{fib}(qg) \arrow[r, dashed, "u"] \arrow[d, dashed, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, dashed] \\
c \arrow[r, "g"'] & a \arrow[r, dashed, "q"'] \arrow[ur, phantom, "\ulcorner", very near end] & \operatorname{cofib}(f)
\end{tikzcd} \tag{4.4.9}$$

This is followed by a digression "on the use of generalized elements to define functors":

> *The first paragraph of the proof just given takes a generic family of cospans and constructs a rectangular diagram (4.4.9), to which [pullback composition and cancellation] can be applied. By the Yoneda lemma, a construction given as a mapping on generalized elements defines an arrow internally to the ∞-cosmos, in this case taking the form of a functor $A^{\lrcorner} \to A^{3\times 2}$, as we now illustrate by unpacking each of the steps.*[2] "

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[2] Indeed, this functor can be understood as the result of applying the construction to the universal generalized element, which is always given by the identity. The generic cospan $g \vee f \colon X \to A^{\lrcorner}$ is used in place of the universal cospan $\mathrm{id} \colon A^{\lrcorner} \to A^{\lrcorner}$ to introduce some human-readable notation.

The rest of the construction

First, we build, from the generic cospan, the dashed arrow below-left, which forms a diagram that glues this cospan to the cofiber sequence associated to one of its legs:

$$\begin{tikzcd}
& A^{\mathbb{2}} \arrow[dr, "\mathrm{cofib}"] & \\
A^{\lrcorner} \arrow[ur, "\mathrm{res}_{\downarrow}"] \arrow[r, dashed] \arrow[dr, equal] & A^{\lrcorner\boxdot} \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\boxdot} \arrow[d, "\mathrm{res}_{\downarrow}"] \\
& A^{\lrcorner} \arrow[r, "\mathrm{res}_{\downarrow}"'] & A^{\mathbb{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
& & A^{\mathbb{2}} \arrow[dr, "\mathrm{fib}"] & \\
A^{\lrcorner} \arrow[r, dashed] & A^{\lrcorner\boxdot} \arrow[ur, "\mathrm{res}_{\to\to}"] \arrow[r, dashed] \arrow[dr, equal] & A^{\boxdot\boxdot} \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\boxdot} \arrow[d, "\mathrm{res}"] \\
& & A^{\lrcorner\boxdot} \arrow[r, "\mathrm{res}_{\lrcorner}"'] & A^{\lrcorner}
\end{tikzcd}$$

The simplicial set $\lrcorner\boxdot \cong \lrcorner \cup_{\downarrow} \boxdot$ does not include the composite 1-simplex from the lower-left vertex to the lower-right vertex but this can be attached by filling an inner horn, resulting in an equivalent ∞-category that we also denote by $A^{\lrcorner\boxdot}$. Next we attach the fiber sequence associated to that composite arrow, gluing the exterior rectangle onto the diagram of shape $\lrcorner\boxdot$, defining the dashed arrow above-right.

The simplicial set $\boxdot\boxdot$ is a subset of the rectangle diagram shape $\mathbb{3} \times \mathbb{2}$. In the notation of (4.4.9) what is missing is the map u and the left-hand square, which we induce by the universal property of the fiber sequence $b \to a \to \mathrm{cofib}(f)$, encoded by the absolute right lifting diagram below-left. There is a functor $(g, \mathrm{id}, \mathrm{id})\colon \lrcorner \times \mathbb{2} \to \boxdot\boxdot$ inducing the natural transformation γ below-center, which then factors as below-right:

$$\begin{tikzcd}
& A^{\boxdot} \arrow[d, "\mathrm{res}"] \\
A^{\boxdot\boxdot} \arrow[ur, "\mathrm{res}_{\boxdot}"] \arrow[r, "\mathrm{res}_{\lrcorner}"'] \arrow[r, phantom, shift left=3, "="] & A^{\lrcorner}
\end{tikzcd}
\qquad
\begin{tikzcd}
A^{\boxdot\boxdot} \arrow[r, "\mathrm{res}_{\boxdot}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\gamma"] & A^{\boxdot} \arrow[d, "\mathrm{res}"] \\
A^{\boxdot\boxdot} \arrow[r, "\mathrm{res}_{\lrcorner}"'] & A^{\lrcorner}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A^{\boxdot\boxdot} \arrow[r, "\mathrm{res}_{\boxdot}"] \arrow[d, equal] \arrow[r, phantom, shift right=3, "\Downarrow v"] & A^{\boxdot} \arrow[d, "\mathrm{res}"] \\
A^{\boxdot\boxdot} \arrow[ur, "\mathrm{res}_{\boxdot}"] \arrow[r, "\mathrm{res}_{\lrcorner}"'] \arrow[r, phantom, shift left=3, "="] & A^{\lrcorner}
\end{tikzcd}$$

The composite functor

$$A^{\lrcorner} \longrightarrow A^{\boxdot\boxdot} \xrightarrow{\ulcorner v \urcorner} A^{\boxdot \times \mathbb{2}} \overset{\mathrm{res}}{\twoheadrightarrow} A^{\mathbb{3}\times\mathbb{2}}$$

builds the diagram on display in (4.4.9) from a generic cospan.

Prospects for formalization?

I can imagine three strategies for formalizing the above proof, and the background mathematics upon which it depends.

Strategy I. Given precise definitions of initial and terminal object and fiber and cofiber in the quasi-categorical model. Prove pullback composition and cancellation and that "universal properties are pointwise defined" to avoid the need for generalized elements.

Strategy II. Axiomatize the $(\infty, 2)$-category of ∞-categories using the notion of ∞-cosmos or something similar. Use the definitions and properties of initial and terminal object and fiber and cofiber for generalized elements in an ∞-cosmos. To show that this theory is non-vacuous, prove the quasi-categories define an ∞-cosmos (and formalize other examples, as desired).

Strategy III. Avoid the technicalities of set-based models by developing the theory of ∞-categories synthetically, in a domain-specific type theory. Here generalized elements will just be terms in a context, and all constructions are pointwise defined. Formalization then requires a bespoke proof assistant such as `Rzk`.

References

Papers:

- Emily Riehl, Could ∞-category theory be taught to undergraduates?, Notices of the AMS 70(5):727–736, May 2023; arXiv:2302.07855
- Nikolai Kudasov, Emily Riehl, Jonathan Weinberger, Formalizing the ∞-categorical Yoneda lemma, CPP 2024: 274–290; arXiv:2309.08340

Formalization:

- Johan Commelin, Kim Morrison, Joël Riou, Adam Topaz, a nascent theory of quasi-categories in mathlib, Mathlib/AlgebraicTopology/Quasicategory.lean
- Mario Carneiro and Emily Riehl, work in progress towards a model-independent theory, mathlib4/tree/infty-cosmos
- Nikolai Kudasev et al, synthetic ∞-categories in simplicial homotopy type theory, rzk-lang.github.io/sHoTT/

Danke!