Emily Riehl

Johns Hopkins University

Prospects for Computer Formalization of Infinite-Dimensional Category Theory

UCLouvain, with the support of the Hoover Foundation

Plan

1. Anecdotes
   - The future of curiosity-driven research
   - An experimental formalization challenge

2. Challenges
   - Formalizing post-rigorous mathematics
   - The post-rigorous literature on ∞-categories

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Anecdotes

An attempted proof of Grothendieck's homotopy hypothesis

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES
VOL. XXXII-1 (1991)

∞-GROUPOIDS AND HOMOTOPY TYPES

by M.M. KAPRANOV and V.A. VOEVODSKY

RÉSUMÉ. Nous présentons une description de la categorie homotopique des CW-complexes en termes des ∞-groupoïdes. La possibilité d'une telle description a été suggérée par A. Grothendieck dans son mémoire "A la poursuite des champs".

It is well-known [GZ] that CW-complexes X such that $\pi_i(X,x) = 0$ for all $i \geq 2$, $x \in X$, are described, at the homotopy level, by groupoids. A. Grothendieck suggested, in his unpublished memoir [Gr], that this connection should have a higher-dimensional generalisation involving polycategories. viz. polycategorical analogues of groupoids. It is the purpose of this paper to establish such a generalisation.

- 15 statements =
  4 theorems
  \+ 9 propositions
  \+ 1 lemma
  \+ 1 corollary
- 5 short "obvious" proofs + 3 proofs

- Carlos Simpson's "Homotopy types of strict 3-groupoids" (1998) shows that the 3-type of S^2 can't be realized by a strict 3-groupoid — contradicting the last corollary
- But no explicit mistake was found. Voevodsky: "I was sure that we were right until the fall of 2013 (!!)"

A sociological problem

MATHEMATICS

The Origins and Motivations of Univalent Foundations

A Personal Mission to Develop Computer Proof Verification to Avoid Mathematical Mistakes

By Vladimir Voevodsky • Published 2014

"A technical argument by a trusted author, which is hard to check and looks similar to arguments known to be correct, is hardly ever checked in detail."

Voevodsky on the future of "curiosity-driven research"

Around the time that I discovered the mistake in my motivic paper, I was working on a new development, which I called 2-theories... The mathematics of 2-theories is an example of precisely that kind of higher-dimensional mathematics that Kapranov and I had dreamed about in 1989. And I really enjoyed discovering new structures that were not direct extensions of structures in lower dimensions.

But to do the work at the level of rigor and precision I felt was necessary would take an enormous amount of effort and would produce a text that would be very hard to read. And who would ensure that I did not forget something and did not make a mistake, if even the mistakes in much more simple arguments take years to uncover? I think it was at this moment that I largely stopped doing what is called "curiosity-driven research" and started to think seriously about the future. I didn't have the tools to explore the areas where curiosity was leading me and the areas that I considered to be of value and of interest and of beauty.

Voevodsky on practical foundations for computer proof assistants

So I started to look into what I could do to create such tools. And it soon became clear that the only long-term solution was somehow to make it possible for me to use computers to verify my abstract, logical, and mathematical constructions. The software for doing this has been in development since the sixties...but none of them was in any way appropriate for the kind of mathematics for which I needed a system.

The primary challenge that needed to be addressed was that the foundations of mathematics were unprepared for the requirements of the task. Formulating mathematical reasoning in a language precise enough for a computer to follow meant using a foundational system of mathematics not as a standard of consistency to establish a few fundamental theorems, but as a tool that can be employed in everyday mathematical work. There were two main problems with the existing foundational systems, which made them inadequate...

Voevodsky in 2014

And I now do my mathematics with a proof assistant. I have a lot of wishes in terms of getting this proof assistant to work better, but at least I don't have to go home and worry about having made a mistake in my work. I know that if I did something, I did it, and I don't have to come back to it nor do I have to worry about my arguments being too complicated or about how to convince others that my arguments are correct. I can just trust the computer. There are many people in computer science who are contributing to our program, but most mathematicians still don't believe that it is a good idea. And I think that is very wrong.

The Liquid Tensor Experiment

In December 2020, Peter Scholze announced the Liquid Tensor Experiment in a guest post on a blog run by Kevin Buzzard, an algebraic number theorist and active user of the Lean computer proof assistant.

1. The challenge

I want to propose a challenge: Formalize the proof of the following theorem.

Theorem 1.1 *(Clausen-S.) Let $0 < p' < p \leq 1$ be real numbers, let S be a profinite set, and let V be a p-Banach space. Let $\mathcal{M}_{p'}(S)$ be the space of p'-measures on S. Then*

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geq 1$.

(This is a special case of Theorem 9.1 in www.math.uni-bonn.de/people/scholze/Analytic.pdf, and is the essence of the proof of Theorem 6.5 there.)

Why formalize this?

After explaining the main mathematical ideas, Scholze continues:

6. Sympathy for the devil

Why do I want a formalization?

— I spent much of 2019 obsessed with the proof of this theorem, almost getting crazy over it. In the end, we were able to get an argument pinned down on paper, but I think nobody else has dared to look at the details of this, and so I still have some small lingering doubts.

— as I explain below, the proof of the theorem has some very unexpected features. In particular, it is very much of *arithmetic* nature. It is the kind of argument that needs to be closely inspected.

— while I was very happy to see many study groups on condensed mathematics throughout the world, to my knowledge all of them have stopped short of this proof. (Yes, this proof is not much fun…)

Testing Lean's Mathlib

— the Lean community has already showed some interest in formalizing parts of condensed mathematics, so the theorem seems like a good goalpost.

— from what I hear, it sounds like the goal is not completely out of reach. (Besides some general topos theory and homological algebra (and, for one point, a bit of stable homotopy theory(!)), the argument mostly uses undergraduate mathematics.) If achieved, it would be a strong signal that a computer verification of current research in very abstract mathematics has become possible. I'll certainly be excited to watch any progress.

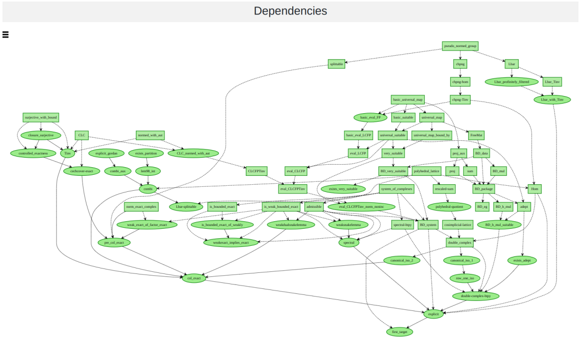
— I think this may be my most important theorem to date. (It does not really have any applications so far, but I'm sure this will change.) Better be sure it's correct…

Can computer proof assistants help mathematicians understand their work in real time?

What happened next

Six months later, Scholze reported:

> *While this challenge has not been completed yet, I am excited to announce that the Experiment has verified the entire part of the argument that I was unsure about. I find it absolutely insane that interactive proof assistants are now at the level that within a very reasonable time span they can formally verify difficult original research.*



The full experiment was completed a little more than a year later.

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Challenges

Pre-rigorous, rigorous, and post-rigorous mathematics

The phrase post-rigorous mathematics refers to Terry Tao's blog post "There's more to mathematics than rigour and proofs":

> *One can roughly divide mathematical education into three stages:*
>
> 1. *The "pre-rigorous" stage, in which mathematics is taught in an informal, intuitive manner, based on examples, fuzzy notions, and hand-waving. …The emphasis is more on computation than on theory. This stage generally lasts until the early undergraduate years.*
> 2. *The "rigorous" stage, in which one is now taught that in order to do maths "properly", one needs to work and think in a much more precise and formal manner …The emphasis is now primarily on theory; and one is expected to be able to comfortably manipulate abstract mathematical objects without focusing too much on what such objects actually "mean". This stage usually occupies the later undergraduate and early graduate years.*
> 3. *The "post-rigorous"stage, in which one has grown comfortable with all the rigorous foundations of one's chosen field, and is now ready to revisit and refine one's pre-rigorous intuition on the subject, but this time with the intuition solidly buttressed by rigorous theory. … The emphasis is now on applications, intuition, and the "big picture". This stage usually occupies the late graduate years and beyond.*

Post-rigorous mathematics

The point of rigour is not to destroy all intuition; instead, it should be used to destroy bad intuition while clarifying and elevating good intuition. It is only with a combination of both rigorous formalism and good intuition that one can tackle complex mathematical problems; one needs the former to correctly deal with the fine details, and the latter to correctly deal with the big picture. Without one or the other, you will spend a lot of time blundering around in the dark (which can be instructive, but is highly inefficient). So once you are fully comfortable with rigorous mathematical thinking, you should revisit your intuitions on the subject and use your new thinking skills to test and refine these intuitions rather than discard them. ...

The ideal state to reach is when every heuristic argument naturally suggests its rigorous counterpart, and vice versa. Then you will be able to tackle maths problems by using both halves of your brain at once — i.e., the same way you already tackle problems in "real life".

— *Terry Tao*

Post-rigorous mathematics in practice

Unfortunately, mathematicians often fail to operate in Tao's post-rigorous ideal state. In practice, a proof might be called "post-rigorous" if:

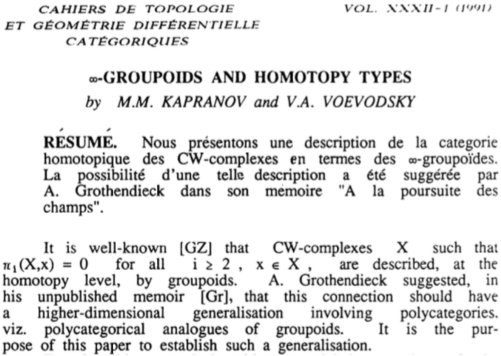
- The argument is not explained in full detail.
- Some claims made as part of the argument may not quite be true as stated.
- Nevertheless, the proof is "morally correct."

Tao continuous:

> *It is perhaps worth noting that mathematicians at all three of the above stages of mathematical development can still make formal mistakes in their mathematical writing. However, the nature of these mistakes tends to be rather different, depending on what stage one is at … [and] can lead to the phenomenon (which can often be quite puzzling to readers at earlier stages of mathematical development) of a mathematical argument by a post-rigorous mathematician which locally contains a number of typos and other formal errors, but is globally quite sound, with the local errors propagating for a while before being cancelled out by other local errors.*

Case studies from the literature

The literature developing the theory of $(\infty, 1)$-categories — commonly nicknamed "∞-categories" — is arguably post-rigorous.



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A few papers contain proofs that aren't so much "post-rigorous" as simply incorrect.

But more commonly, the literature contains "morally-correct" proofs that rely on incomplete definitions, sketched arguments, or explicit unproven conjectures.

Avoiding a precise definition of ∞-categories

The precursor to Jacob Lurie's *Higher Topos Theory* is a 2003 preprint On ∞-Topoi, which avoids using a precise definition of ∞-categories[1]:

> *We will begin in §1 with an informal review of the theory of ∞-categories. There are many approaches to the foundation of this subject, each having its own particular merits and demerits. Rather than single out one of those foundations here, we shall attempt to explain the ideas involved and how to work with them. The hope is that this will render this paper readable to a wider audience, while experts will be able to fill in the details missing from our exposition in whatever framework they happen to prefer.*

Perlocutions of this form are quite common in the field — however the book *Higher Topos Theory* does not proceed in this manner, instead proving theorems for a concrete model of ∞-categories.

[1] In the parlance of the field, selecting a set-theoretic definition of ∞-categories is referred to as "choosing a model."

A proof(?) of the cobordism hypothesis

The cobordism hypothesis classifies (fully-extended) topological quantum field theories, which are functors indexed by a suitably-defined higher category of cobordisms between framed n-manifolds with corners. In a celebrated expository article on the subject, Dan Freed writes:

> *The cobordism hypothesis was conjectured by Baez-Dolan in the mid 1990s. It has now been proved by Hopkins-Lurie in dimension two and by Lurie in higher dimensions. There are many complicated foundational issues which lie behind the definitions and the proof, and only a detailed sketch has appeared so far.*[1]

The footnote elaborates:

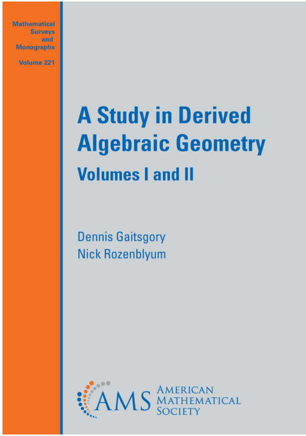
> [1] *Nonetheless, we use "theorem" and its synonyms in this manuscript. The foundations are rapidly being filled in and alternative proofs have also been carried out, though none has yet appeared in print.*

There seems to be no clear consensus on this point of view: a mathOVERFLOW question "What is the status of the cobordism hypothesis?" asked a bit over a year ago remains open.

A conjectural(?) study in derived algebraic geometry

A two-volume study in derived algebraic geometry runs to nearly 1000 pages. Much of the first volume is devoted to developing necessary preliminary results in $(\infty, 1)$-category theory and $(\infty, 2)$-category theory, and includes the following disclaimer:

> *Unfortunately, the existing literature on* $(\infty, 2)$*-categories does not contain the proofs of all the statements that we need. We decided to leave some of the statements unproved, and supply the corresponding proofs elsewhere (including the proofs here would have altered the order of the exposition, and would have come at the expense of clarity).*



This is followed by a list of seven unproved statements.

Obstructions to formalization?

How might this literature read differently in a future where mathematicians are expected to work interactively with a computer proof assistant?

- An incorrect proof should not be formalizable — which is of course a good thing. And perhaps the process of formalization would help identify the error by calling attention to a subtle obstacle to be overcome.
- If it is undesirable to give a precise construction of a mathematical notion (eg of the $(\infty, 2)$-category of ∞-categories), one could instead axiomatize the necessary properties (and hope that the theory is not vacuous).
- Sketch proofs will be harder to implement, as a proof assistant will require clearer definitions and scaffolding. But a formalized sketch, will make it much clearer what gaps remain in the proof.
- A proof modulo unproven conjectures should be formalizable, provided those conjectures and clearly stated in exactly the way they are used.

Prospects for formalization?

I can imagine three strategies for formalizing the theory of ∞-categories.

Strategy I. Given precise definitions of ∞-categorical notions in the quasi-categorical model. Prove theorems using the combinatorics of that model.

Strategy II. Axiomatize the $(\infty, 2)$-category of ∞-categories using the notion of ∞-cosmos or something similar. State and prove theorems about ∞-categories in the axiomatic language of an ∞-cosmos and its quotient 2-category. To show that this theory is non-vacuous, prove the quasi-categories define an ∞-cosmos (and formalize other examples, as desired).

Strategy III. Avoid the technicalities of set-based models by developing the theory of ∞-categories synthetically, in a domain-specific type theory. In simplicial homotopy type theory, an ∞-category can be defined to be a type with unique binary composition of arrows in which paths are equivalent to isomorphisms. Formalization then requires a bespoke proof assistant such as `Rzk`.

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- Mario Carneiro, Emily Riehl, and Dominic Verity, a blueprint of the model-independent theory, emilyriehl.github.io/infinity-cosmos
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Merci!