Emily Riehl

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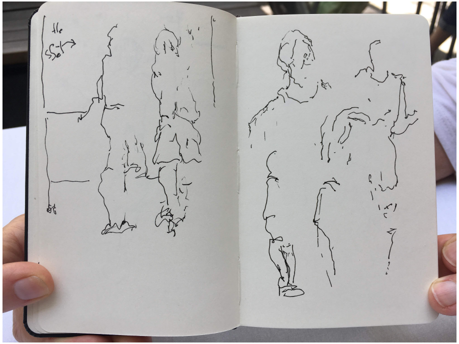
The mathematical multiverse: the case against a unified mathematical reality

Investigating Reality: A Philosophical, Mathematical, and Scientific Exploration

My platonist view of mathematical reality



My structuralist view of mathematical reality

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- Zermelo: $\mathbb{N}_Z := \{0 := \{\}, 1 := \{\{\}\}, 2 := \{\{\{\}\}\}, 3 := \{\{\{\{\}\}\}\}, \ldots\}$
- Von Neumann: $\mathbb{N}_{vN} := \{0 := \{\}, 1 := \{\{\}\}, 2 := \{\{\},\{\{\}\}\}, 3 := \{\{\},\{\{\}\},\{\{\},\{\{\}\}\}\}, \ldots\}$

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Dedekind's Categoricity Theorem All triples given by a set $\mathbb{N}$, an element $0 \in \mathbb{N}$, and a function $\mathsf{succ} : \mathbb{N} \to \mathbb{N}$ satisfying Peano's postulates are isomorphic.

- 0 is not the successor of any natural number.
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Q: Is 3 an element of 17? — Paul Benacerraf “What numbers could not be”

Creation vs discovery

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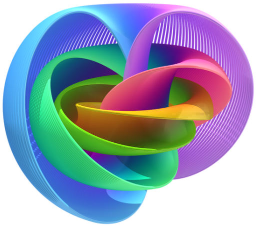

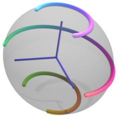

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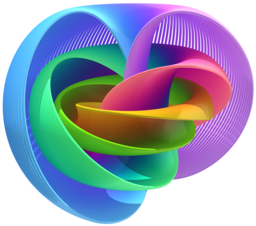

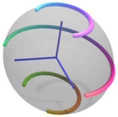

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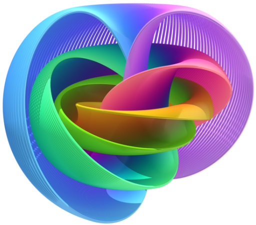

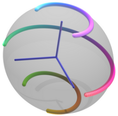

In 1953, Serre discovered that for the vast majority of dimensions this count is finite. Algebraic topologists then created chromatic homotopy theory to discover properties about these functions in the dimensions where the full count is still unknown.

1

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Q: Are these foundations universal? — No, for two orthogonal reasons.

- Certain mathematical questions cannot be resolved without additional axioms, e.g., assuming the existence of particular "large" sets.
- The axioms of first order logic and set theory with choice are not universally true!

The axiom of choice

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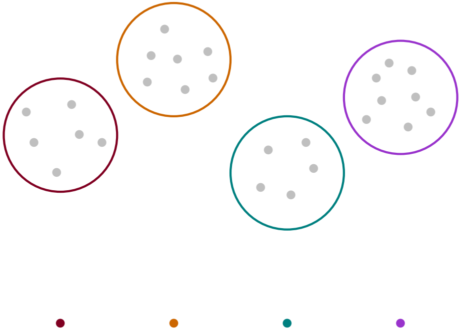

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“The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn’s lemma?” — Jerry Bona

Mirrored sets

The axiom of choice is false for mirrored sets — sets with a reflection — because it may be impossible to make a choice that is compatible with the reflection.

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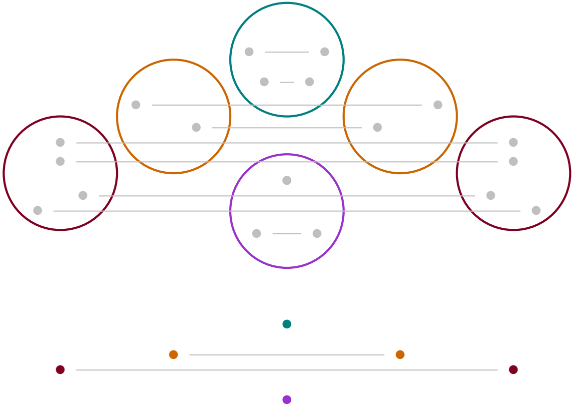

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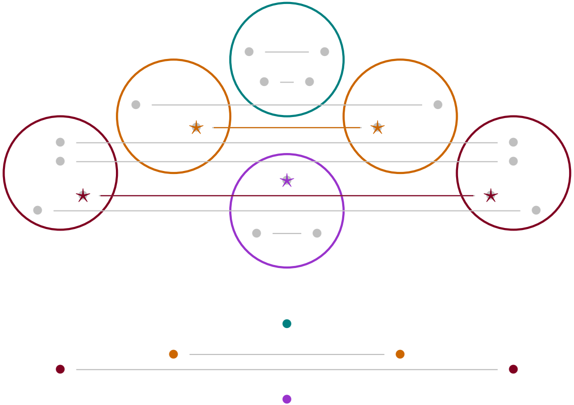

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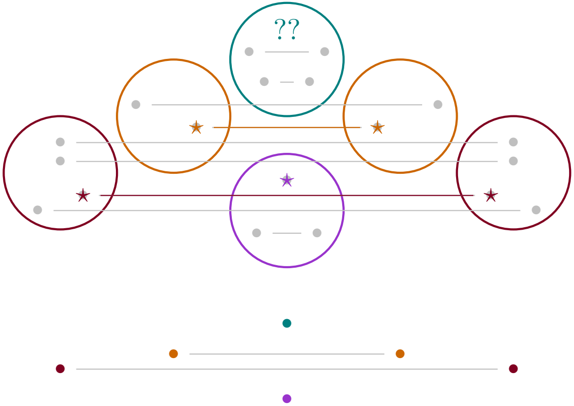

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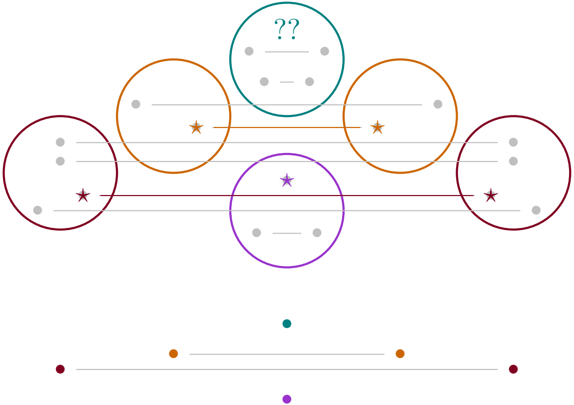

[1]While this demonstrates that mirrored sets do not satisfy the external axiom of choice, as a boolean topos, they do satisfy the internal axiom of choice.

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Mathematics is at its most powerful when it's possible to exercise choice about choice, adopting it in certain domains while rejecting it in others.

2

The mathematical multiverse

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An alternate perspective on foundations is provided by internal logics, which provide domain-specific formal systems for particular mathematical "realities."

- classical first order logic
- linear logic
- coherent or geometric logic
- extensional dependent type theory
- homotopy type theory/univalent foundations
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  E.g.: "by path induction, since equality is reflexive it is also symmetric" is a complete proof in homotopy type theory, but meaningless in other settings.

Exploring the mathematical multiverse

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- The mathematician inputs each line of their proof in a precise syntax.
- The computer verifies the logical reasoning produces a valid deduction of the claimed mathematical statement.
- Depending on the sophistication of the computer program, it might also “assist” the mathematician in various ways:
  - By catching errors of reasoning (unjustified assumptions, missing cases, etc).
  - By keeping track of where they are in a complicated logical argument.
  - By suggesting or even automatically generating proofs (“auto-formalization”).

Computer formalization of mathematics

> *Formalized mathematics, in tandem with other forms of computerized mathematics*[2]*, provides better management of mathematical knowledge, an opportunity to carry out ever more complex and larger projects, and hitherto unseen levels of precision.*
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> — *Andrej Bauer, "The dawn of formalized mathematics," delivered at the 8th European Congress of Mathematics*

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Recent successes include:

- the Kepler conjecture, resolving a 1611 conjecture, 2003–2014, HOL Light
- the Feit-Thompson Odd Order Theorem, a foundational result in the classification of finite simple groups, 2006–2012, Coq
- the liquid tensor experiment, formalizing condensed mathematics, 2020–2022, Lean
- the Brunerie number, computing $\pi_4 S^3 \cong \mathbb{Z}/2\mathbb{Z}$, 2015–2022, Cubical agda

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Could ∞-Category Theory Be Taught to Undergraduates?

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1. The Algebra of Paths

It is natural to probe a suitably nice topological space X by means of its *paths*, the continuous functions from the standard unit interval $I \coloneqq [0,1] \subset \mathbb{R}$ to X. But what structure do the paths in X form?

To start, the paths form the edges of a directed graph whose vertices are the points of X: a path $p : I \to X$ defines an arrow from the point $p(0)$ to the point $p(1)$. Moreover, this graph is *reflexive*, with the constant path refl_x at each point $x \in X$ defining a distinguished endoarrow.

Can this reflexive directed graph be given the structure of a category? To do so, it is natural to define the composite of a path p from x to y and a path q from y to z by gluing together these continuous maps—i.e., by concatenating the paths—and then by reparametrizing via the homeomorphism $I \cong I \cup_{1=0} I$ that traverses each path at double speed:

$$\begin{tikzcd} I \arrow[r, "\cong"] \arrow[rr, bend right, "p*q"'] & I \cup_{1=0} I \arrow[r, "p \cup q"] & X \end{tikzcd} \tag{1.1}$$

But the composition operation $*$ fails to be associative or unital. In general, given a path r from z to w, the

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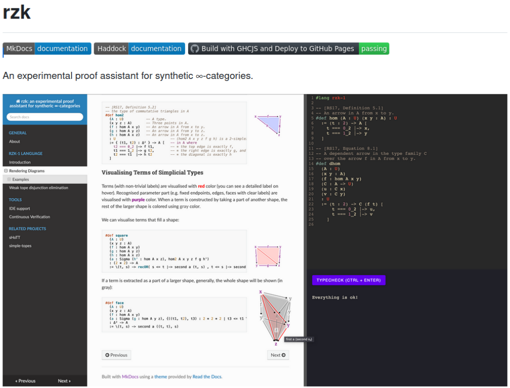

I'd argue that the traditional foundations of mathematics are not really suitable for "higher mathematics" such as ∞-category theory, where the basic objects are built out of higher-dimensional types instead of mere sets.

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