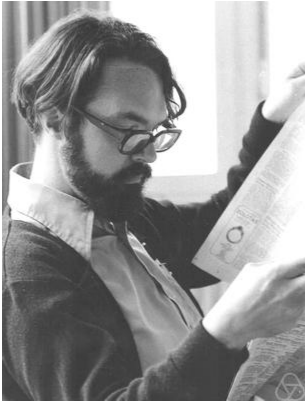
Johns Hopkins University

Emily Riehl

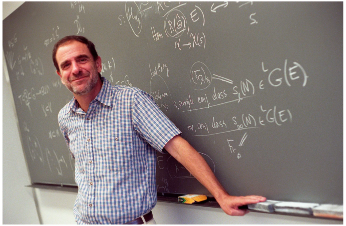
Synthetic Perspectives on Higher Structures

International Congress of Mathematicians

Dedication



E. Graham Evans Jr
1942–2021



Benedict Gross
1950–2025

Jack Morava
1944–2025

Collaborators

Dominic Verity

Nikolai Kudasov

Evan Cavallo

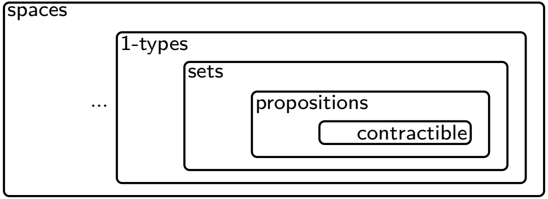
Michael Shulman

Jonathan Weinberger

Christian Sattler

On “higher structures”

Nicolas Bourbaki famously based the *Éléments de mathématique* on an abstract mathematical notion of structure: sets, with functions and relations, satisfying axioms.



We use “higher structures” to refer to structures built from spaces/homotopy types/anima instead of sets.

Example: (ordinary) categories involve

- a set of objects C
- sets of arrows $(\mathrm{Hom}(x,y))_{x,y:C}$
- a composition function, etc.

Example: (higher) categories involve

- a space of objects C
- spaces of arrows $(\mathrm{Hom}(x,y))_{x,y:C}$
- weak composition, etc.

In more standard terminology: (higher) categories := ∞-categories := $(\infty, 1)$-categories.

Reimagining the foundations of higher category theory

III. Synthetic (Riehl–Shulman, Weaver–Licata, Gratzer–Weinberger–Buchholtz, Riva, Cnossen–Kobe...)

II. Axiomatic (Toën, Barwick–Schommer-Pries, Riehl–Verity, Cisinski–Cnossen–Nguyen–Walde, ...)

I. Analytic (Boardman–Vogt, Dwyer–Kan, Hirschowitz–Simpson, Rezk, Pellissier, Bergner, Joyal, Lurie, Joyal–Tierney, Verity, Simpson, Barwick–Kan, Gepner–Haugseng, ...)

"analytic" :: $\mathbb{R}$ defined using Cauchy sequences or Dedekind cuts
"axiomatic" :: $\mathbb{R}$ as a complete ordered field
"synthetic" :: non-standard analysis in internal set theory

Computer formalization of synthetic homotopy theory

Motivation: synthetic proofs/constructions are more easily formalized.

Synthetic spaces via homotopy type theory (in Agda, Cubical Agda, Lean2, Rocq)
Blakers–Massey theorem, coherent idempotents, cohomology rings with computations, Eckmann–Hilton argument, Eilenberg–MacLane spaces, Eilenberg–Steenrod axioms, Freudenthal suspension theorem, (central) H-spaces, hairy ball theorem, Hilton–Milnor theorem, Hurewicz theorem, localizations, real projective spaces, Serre finiteness theorem, Serre spectral sequence, the syllepsis, Whitehead products, $\pi_3 S^2$, $\pi_4 S^3$, ...

Analytic spaces in Mathlib (in Lean4)

- Eckmann–Hilton argument
- definition and calculations of a few trivial homotopy groups
- singular homology with calculations for totally disconnected spaces

Plan

1. Synthetic perspectives on spaces (via homotopy type theory)

2. Synthetic perspectives on categories (via simplicial type theory)

1

Synthetic perspectives on spaces (via homotopy type theory)

Families of spaces

Write $(A_\gamma)_{\gamma:\Gamma}$ for a "Γ-indexed family of spaces" or a "space A in context Γ."

Here $\gamma : \Gamma$ is a (generalized) element of the base space Γ and A_γ is the fiber:

$$\begin{array}{ccc} A_\gamma & \longrightarrow & A \\ \downarrow\!\!\downarrow & \lrcorner & \downarrow\!\!\downarrow (A_\gamma)_{\gamma:\Gamma} \\ \Delta & \xrightarrow[\gamma]{} & \Gamma \end{array}$$

Example: Any space A has a path space family $A \overset{\text{refl}}{\underset{\sim}{\rightarrowtail}} A^I \xrightarrow{(\text{ev}_0,\text{ev}_1)} \!\!\!\rightarrow A \times A$ defined by exponentiation with the interval $* + * \rightarrowtail I \overset{\sim}{\twoheadrightarrow} *$.

$$\begin{array}{ccc} x \sim y & \longrightarrow & A^I \\ \downarrow\!\!\downarrow & \lrcorner & \downarrow\!\!\downarrow (x{\sim}y)_{x,y:A} \\ * & \xrightarrow[(x,y)]{} & A \times A \end{array}$$

In this family, the fiber $x \sim y$ over two points is the space of paths in A from x to y.

A very convenient category of spaces

The category of spaces is locally cartesian closed: any $f\colon \Delta \to \Gamma$ defines an adjoint triple of operations on families

$$\mathsf{Spaces}_{/\Delta} \;\underset{\Pi_f}{\overset{\Sigma_f}{\rightleftarrows}}\; \mathsf{Spaces}_{/\Gamma}, \qquad \Sigma_f \dashv f^* \dashv \Pi_f, \quad f^*\colon \mathsf{Spaces}_{/\Gamma} \to \mathsf{Spaces}_{/\Delta}$$

acting by

composition / sum:

$$\begin{array}{ccc} A & = & \Sigma_{\gamma:\Gamma} A_\gamma \\ {\scriptstyle (A_\gamma)_{\gamma:\Gamma}} \downarrow & \overset{\Sigma}{\mapsto} & \downarrow \\ \Gamma & \xrightarrow{\;!\;} & * \end{array}$$

pushforward / sections:

$$\begin{array}{ccc} A & & \Pi_{\gamma:\Gamma} A_\gamma \\ {\scriptstyle (A_\gamma)_{\gamma:\Gamma}} \downarrow & \overset{\Pi}{\mapsto} & \downarrow \\ \Gamma & \xrightarrow{\;!\;} & * \end{array}$$

pullback / substitution:

$$\begin{array}{ccc} A_f & \longrightarrow & A \\ {\scriptstyle (A_{f(\delta)})_{\delta:\Delta}} \downarrow & \overset{f^*}{\leftarrow\!\shortmid} & \downarrow {\scriptstyle (A_\gamma)_{\gamma:\Gamma}} \\ \Delta & \xrightarrow{\;f\;} & \Gamma \end{array}$$

with all operations defined fiberwise.

A point in $\Sigma_{\gamma:\Gamma} A_\gamma$ choses a point in a particular fiber—"$\exists_{\gamma:\Gamma}, A_\gamma$."

A point in $\Pi_{\gamma:\Gamma} A_\gamma$ continuously chooses elements in every fiber—"$\forall_{\gamma:\Gamma}, A_\gamma$."

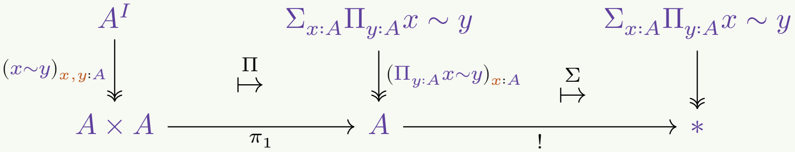
Propositions as spaces

Using these operations, we may define moduli spaces for various propositions.

Example: For any space A, a point in the space

$$\mathsf{isContr}(A) :\equiv \Sigma_{x:A}\Pi_{y:A}\, x \sim y$$

provides a proof of the contractibility of A.



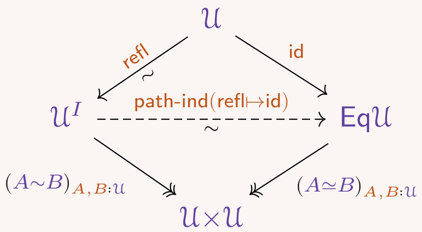
A similar moduli space detects whether a map $f : A \to B$ is an equivalence

$$\mathsf{isEquiv}(f) :\equiv \left(\Sigma_{g:B\to A}\, g \circ f \sim \mathsf{id}_A\right) \times \left(\Sigma_{h:B\to A}\, f \circ h \sim \mathsf{id}_B\right)$$

Univalent universes of spaces

Arbitrary equivalences between arbitrary types are elements of a family of spaces $A \simeq B := \Sigma_{f:A\to B}\text{isEquiv}(f)$ indexed by $A, B : \mathcal{U}$ in the universe of (small) spaces.

Univalence (Voevodsky): Paths in the universe are equivalent to equivalences:



By the principle of path-induction, to define a map out of the total path space $\mathcal{U}^I$ into a family of spaces over $\mathcal{U} \times \mathcal{U}$, it suffices to define its image on the subspace refl : $\mathcal{U} \xrightarrow{\sim} \mathcal{U}^I$ of constant paths.

Application of univalence: (higher) group theory

The structure-identity principle, a consequence of the univalence axiom, characterizes the path spaces of other moduli spaces constructed from the universe.

Example: For a natural number n, $\mathsf{Fin}_n := \Sigma_{X:\mathcal{U}} \| \underline{n} \sim X \|$ is the subuniverse of spaces that are merely equivalent to the discrete space $\underline{n}$ on n points. By univalence:

- Fin_n is a 1-type, with all homotopy groups vanishing above dimension 1.
- Paths $p : X \sim Y$ in Fin_n are equivalent to equivalences $e : X \simeq Y$.
- The based loop space $\Omega(\mathsf{Fin}_n, \underline{n}) := \underline{n} \sim \underline{n}$ is equivalent to $\underline{n} \simeq \underline{n}$.

Consequently $B\Sigma_n := \mathsf{Fin}_n := \Sigma_{X:\mathcal{U}} \| \underline{n} \sim X \|$.

Higher groups are defined analogously for arbitrary $a : A$ by $B\mathsf{Aut}(a) := \Sigma_{x:A} \| a \sim x \|$.

An action of a higher group BG is a map $X : BG \to \mathcal{U}$, defining a family $(X_z)_{z:BG}$, with invariants and coinvariants defined by $X^{hG} := \Pi_{z:BG} X_z$ and $X_{hG} := \Sigma_{z:BG} X_z$.

2

Synthetic perspectives on categories (via simplicial type theory)

Simplices from a strict interval (after Joyal)

Synthetic category theory is developed in the presence of a strict directed interval $\mathbb{2}$:

- with endpoints $0, 1 : \mathbb{2}$,
- with a family of propositions $(x \leq y)_{x,y:\mathbb{2}}$, and
- satisfying reflexivity, transitivity, antisymmetry, totality, with min 0 and max 1.

From the interval, we define the standard simplices (and maps between them):

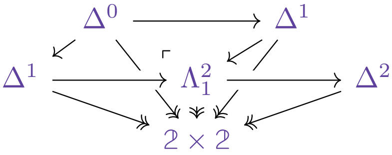
$$\Delta^0 := * \qquad \Delta^2 := \Sigma_{x,y:\mathbb{2}} y \leq x$$

$$\Delta^1 := \mathbb{2} \qquad \Delta^3 := \Sigma_{x,y,z:\mathbb{2}} z \leq y \leq x$$

$$\Delta^2 \xrightarrow{(y \leq x)_{x,y:\mathbb{2}}} \mathbb{2} \times \mathbb{2} \qquad \Delta^3 \xrightarrow{(z \leq y \leq x)_{x,y,z:\mathbb{2}}} \mathbb{2} \times \mathbb{2} \times \mathbb{2}$$

as well as simplicial subspaces such as

$$\Lambda^2_1 := \Sigma_{x,y:\mathbb{2}} (y \sim 0) \vee (x \sim 1)$$



Precategories

Synthetic spaces C in the presence of 2 may be thought of as "simplicial spaces" and have families of paths

$$\begin{array}{c} C^I \\ \downarrow\!\!\!\downarrow (x \sim y)_{x,y:C} \\ C \times C \end{array}$$

and arrows

$$\begin{array}{c} C^{\Delta^1} \\ \downarrow\!\!\!\downarrow (\mathrm{Hom}(x,y))_{x,y:C} \\ C \times C \end{array}$$

with fibers:

$$\begin{array}{ccc} x \sim y & \longrightarrow & C^I \\ \downarrow\!\!\!\downarrow & \lrcorner & \downarrow\!\!\!\downarrow (x\sim y)_{x,y:C} \\ * & \xrightarrow[(x,y)]{} & C \times C \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Hom}(x,y) & \longrightarrow & C^{\Delta^1} \\ \downarrow\!\!\!\downarrow & \lrcorner & \downarrow\!\!\!\downarrow (\mathrm{Hom}(x,y))_{x,y:C} \\ * & \xrightarrow[(x,y)]{} & C \times C \end{array}$$

Definition (Riehl–Shulman): A precategory is a simplicial space C in which every composable pair of arrows has a unique composite:

for all $f : \mathrm{Hom}(x,y)$ and $g : \mathrm{Hom}(y,z)$ the fibers of

$$\begin{array}{c} C^{\Delta^2} \\ \downarrow\!\!\!\downarrow (\mathrm{Comp}(f,g))_{\langle f,g\rangle : C^{\Lambda^2_1}} \\ C^{\Lambda^2_1} \end{array}$$

are contractible.

Composition, identities, and associativity

Definition (Riehl–Shulman): A **precategory** is a simplicial space C in which every composable pair of arrows has a unique composite:

for all $f : \mathrm{Hom}(x,y)$ and $g : \mathrm{Hom}(y,z)$ the fibers of

$$\left(C^{\Delta^2} \xrightarrow{} C^{\Lambda^2_1} \right) \qquad (\mathrm{Comp}(f,g))_{\langle f,g\rangle : C^{\Lambda^2_1}}$$

are contractible.

A precategory has a unique composition function $\mathrm{Hom}(y,z) \times \mathrm{Hom}(x,y) \xrightarrow{\circ} \mathrm{Hom}(x,z)$

$$\begin{array}{ccc} C^{\Lambda^2_1} & \xrightarrow{\quad\circ\quad} & C^{\Delta^1} \\ & \searrow \quad \swarrow & \\ & C \times C & \end{array}$$

$$(\Sigma_{y:C} \mathrm{Hom}(y,z) \times \mathrm{Hom}(x,y))_{x,z:C} \qquad (\mathrm{Hom}(x,z))_{x,z:C}$$

that is unital and associative, where identities are defined as constant maps

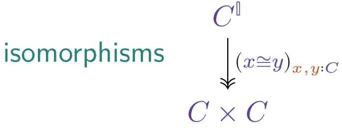
$$\Delta^0 + \Delta^0 \rightarrowtail \Delta^1 \xrightarrow{\sim} \Delta^0 \quad \rightsquigarrow \quad C \xrightarrow{\mathrm{id}} C^{\Delta^1} \xrightarrow{(\mathrm{ev}_{\mathrm{dom}}, \mathrm{ev}_{\mathrm{cod}})} C \times C.$$

Categories

An arrow $f : \mathsf{Hom}(x, y)$ in a precategory is an isomorphism if

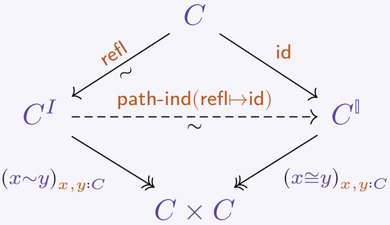
$$\mathsf{isIso}(f) := \left(\Sigma_{g:\mathsf{Hom}(y,x)} g \circ f \sim \mathsf{id}_x\right) \times \left(\Sigma_{h:\mathsf{Hom}(y,x)} f \circ h \sim \mathsf{id}_y\right)$$

Thus, in addition to the families of paths and arrows, precategories have a family of



where $x \cong y := \Sigma_{f:\mathsf{Hom}(x,y)} \mathsf{isIso}(f)$.

Definition (Riehl–Shulman): A precategory C is a category if isomorphisms are equivalent to paths

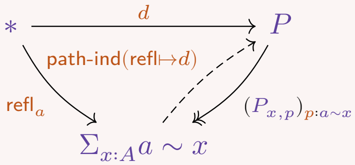


A simplicial space is a groupoid if arrows are equivalent to paths.

Path induction and arrow induction

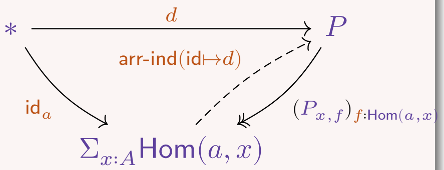
Path Induction (generalizing the indiscernibility of identicals):

For $a : A$, to define a section of a family over the based path space $\Sigma_{x:A} a \sim x$, it suffices to define the image of the constant path refl_a:



Arrow Induction (generalizing the Yoneda lemma):

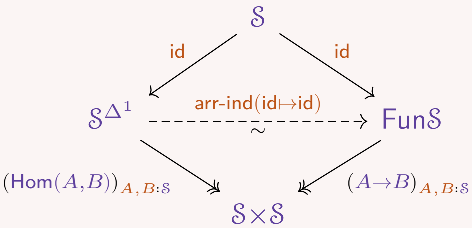
For $a : C$ a precategory, to define a section of a covariant family over the slice precategory $\Sigma_{x:C}\mathsf{Hom}(a, x)$, it suffices to define the image of the identity arrow id_a:



Directed univalence

As univalence characterizes the paths in the universe of spaces,
directed univalence characterizes arrows in the universes of groupoids or categories.

Directed univalence (Weaver–Licata, Cisinksi–Nguyen, Gratzer–Weinberger–Buchholtz, Cavallo–Riehl–Sattler): Arrows in the universe $\mathcal{S}$ are equivalent to functions:



Application of directed univalence

Using the directed univalence axiom, we can construct other categories of higher structures and characterize their hom-spaces, via the structure-homomorphism principle.

Example: The simplicial space of magmas is defined by Magma $:= \Sigma_{A:\mathcal{S}} A \times A \to A$.
By (higher) directed univalence:

- Magma is a category
- an arrow from $(A, \alpha : A \times A \to A)$ to $(B, \beta : B \times B \to B)$ is given by a function $f : A \to B$ with

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow[f]{} & B \end{array}$$

Both results use higher directed univalence,
an equivalence between $\mathcal{S}^{\Delta^n}$ and n composable functions.

Strange new universes of synthetic mathematics

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 2, April 2024, Pages 201–202
https://doi.org/10.1090/bull/1836
Article electronically published on February 20, 2024

WILL MACHINES CHANGE MATHEMATICS?

MAIA FRASER, ANDREW GRANVILLE, MICHAEL H. HARRIS, COLIN MCLARTY,
EMILY RIEHL, AND AKSHAY VENKATESH
(Guest Editorial Committee)

STRANGE NEW UNIVERSES: PROOF ASSISTANTS AND SYNTHETIC FOUNDATIONS

MICHAEL SHULMAN

Abstract. Existing computer programs called proof assistants can verify the correctness of mathematical proofs but their specialized proof languages present a barrier to entry for many mathematicians. Large language models have the potential to lower this barrier, enabling mathematicians to interact with proof assistants in a more familiar vernacular. Among other advantages, this may allow mathematicians to explore radically new kinds of mathematics using an LLM-powered proof assistant to train their intuitions as well as ensure their arguments are correct. Existing proof assistants have already played this role for fields such as homotopy type theory.

"Out of nothing I have created a strange new universe."
– János Bolyai, one of the inventors of non-Euclidean geometry

"I believe that future mathematics software will combine the ease of interaction of an LLM with the near-absolute trustworthiness of a proof assistant.

...just as a present-day mathematician can exploit a computer's calculational ability to explore numerical realms undreamed-of decades ago, these imagined future mathematicians may exploit a computer's logical ability to explore conceptual realms undreamed-of today.

...What new kinds of mathematics, then, may be waiting for us to explore with the proof assistants of tomorrow?"

— Michael Shulman

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