



Emily Riehl

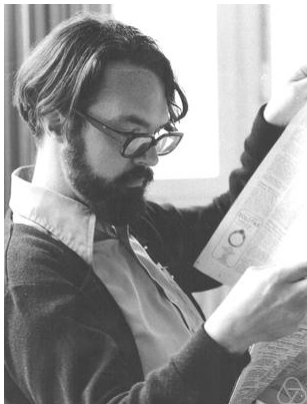


Synthetic Perspectives on Higher Structures

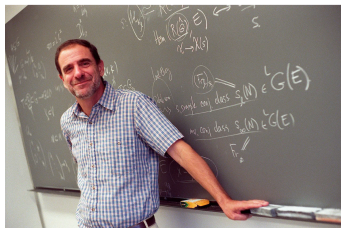


International Congress of Mathematicians

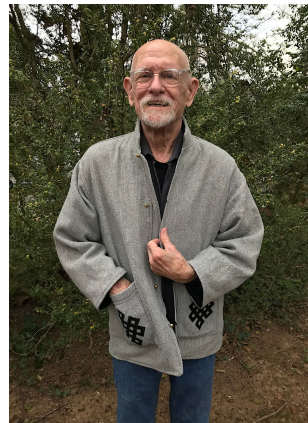
Dedication



E. Graham Evans Jr
1942–2021



Benedict Gross
1950–2025

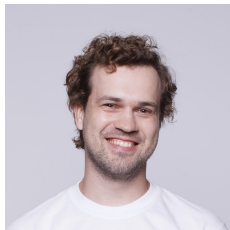


Jack Morava
1944–2025

Collaborators



Dominic Verity



Nikolai Kudasov



Evan Cavallo



Michael Shulman



Jonathan Weinberger



Christian Sattler

On “higher structures”



Nicolas Bourbaki famously based the *Éléments de mathématique* on an abstract mathematical notion of structure: sets, with functions and relations, satisfying axioms.

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We use “higher structures” to refer to analogous objects built from spaces/homotopy types/anima instead of sets.

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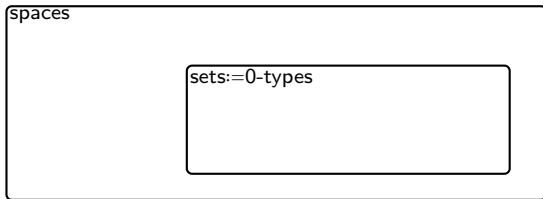
spaces

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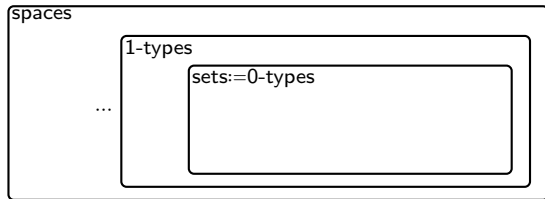


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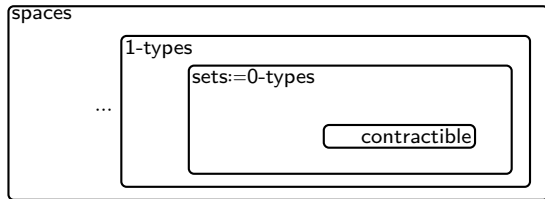


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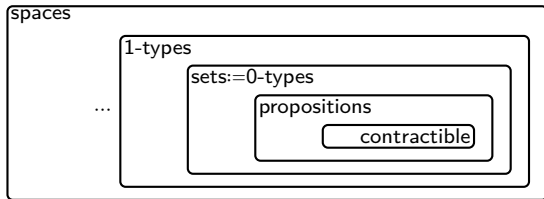


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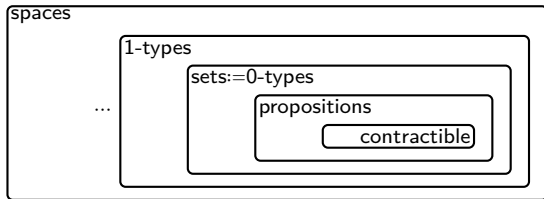


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We use “higher structures” to refer to analogous objects built from spaces/homotopy types/anima instead of sets.

Example: (ordinary) categories involve

- a set of objects C
- sets of arrows $(\text{Hom}(x, y))_{x, y: C}$
- a composition function, etc.

Example: (higher) categories involve

- a space of objects C
- spaces of arrows $(\text{Hom}(x, y))_{x, y: C}$
- weak composition, etc.

In more standard terminology: (higher) categories $:= \infty$ -categories $:= (\infty, 1)$ -categories.



I. Analytic: using concrete “models”

(Boardman–Vogt, Dwyer–Kan, Hirschowitz–Simpson, Rezk, Pellissier, Bergner, Joyal, Lurie, Joyal–Tierney, Verity, Simpson, Barwick–Kan, ...)

“analytic” $\because$ $\mathbb{R}$ defined using Cauchy sequences or Dedekind cuts



II. Axiomatic: aka “model-independent”

(Toën, Barwick–Schommer–Pries, Riehl–Verity, Haugseng, Cisinski–Cnossen–Nguyen–Walde, ...)

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“analytic” :: $\mathbb{R}$ defined using Cauchy sequences or Dedekind cuts

“axiomatic” :: $\mathbb{R}$ as a complete ordered field

Reimagining the foundations of higher category theory



III. Synthetic: in a domain-specific language

(Riehl–Shulman, Weaver–Licata, Buchholtz–Weinberger, Gratzer–Weinberger–Buchholtz, Riva, Cossen–Kobe, ...)

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- “analytic” :: $\mathbb{R}$ defined using Cauchy sequences or Dedekind cuts
- “axiomatic” :: $\mathbb{R}$ as a complete ordered field
- “synthetic” :: non-standard analysis in **internal set theory**

Computer formalization of synthetic homotopy theory



Motivation: synthetic proofs/constructions are more easily formalized.

Synthetic spaces via homotopy type theory (in AGDA, CUBICAL AGDA, LEAN2, ROCQ)
Blakers–Massey theorem, coherent idempotents, cohomology rings with computations, Eckmann–Hilton argument, Eilenberg–MacLane spaces, Eilenberg–Steenrod axioms, Freudenthal suspension theorem, (central) H-spaces, hairy ball theorem, Hilton–Milnor theorem, Hurewicz theorem, localizations, real projective spaces, Serre finiteness theorem, Serre spectral sequence, the syllepsis, Whitehead products, $\pi_3 S^2$, $\pi_4 S^3$, ...

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Analytic spaces in Mathlib (in [LEAN4](#))

- Eckmann–Hilton argument
- definition and calculations of a few trivial homotopy groups
- singular homology with calculations for totally disconnected spaces



1. Synthetic perspectives on spaces (via homotopy type theory)
families of spaces, propositions as spaces, univalent universes
2. Synthetic perspectives on categories (via simplicial type theory)
unique binary composition, path & arrow induction, directed univalent universes



1

Synthetic perspectives on spaces
(via homotopy type theory)

Families of spaces



Write $(A_\gamma)_{\gamma:\Gamma}$ for a “ Γ -indexed family of spaces” or a “space A in context Γ .”

Here $\gamma : \Gamma$ is a (generalized)
element of the base space Γ
and A_γ is the fiber:

$$\begin{array}{ccc} A_\gamma & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow (A_\gamma)_{\gamma:\Gamma} \\ \Delta & \xrightarrow{\gamma} & \Gamma \end{array}$$

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Example: Any space A has a path space family $A \xrightarrow[\sim]{\text{refl}} A^I \xrightarrow{(\text{ev}_0, \text{ev}_1)} A \times A$ defined by exponentiation with the interval $* \vdash * \longrightarrow I \xrightarrow{\sim} *$.

$$\begin{array}{ccc} x \sim y & \longrightarrow & A^I \\ \downarrow & \lrcorner & \downarrow (x \sim y)_{x,y:A} \\ * & \xrightarrow{(x,y)} & A \times A \end{array}$$

In this family, the fiber $x \sim y$ over two points is the space of paths in A from x to y .

A very convenient category of spaces



The category of spaces is **locally cartesian closed**: any $f: \Delta \rightarrow \Gamma$ defines an adjoint

triple of operations on families $\mathbf{Spaces}_{/\Delta} \xleftarrow[\Pi_f]{\Sigma_f} \mathbf{Spaces}_{/\Gamma} \xrightarrow{f^*}$ acting by

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pullback /
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A point in $\Sigma_{\delta:\Delta} B_\delta$ chooses a point in a particular fiber—“ $\exists_{\delta:\Delta}, B_\delta$.”
A point in $\Pi_{\delta:\Delta} B_\delta$ continuously chooses elements in every fiber—“ $\forall_{\delta:\Delta}, B_\delta$.”

Propositions as spaces (and contractibility as uniqueness)



Using these operations, we may define moduli spaces for various propositions.

Example: For any space A , a point in the space

$$\text{isContr}(A) := \sum_{x:A} \prod_{y:A} x \sim y$$

provides a proof of the contractibility of A .

$$\begin{array}{ccccc} A^I & & \sum_{x:A} \prod_{y:A} x \sim y & & \sum_{x:A} \prod_{y:A} x \sim y \\ \downarrow (x \sim y)_{x,y:A} & \xrightarrow[\Pi]{\vdash} & \downarrow (\prod_{y:A} x \sim y)_{x:A} & \xrightarrow[\Sigma]{\vdash} & \downarrow \\ A \times A & \xrightarrow[\pi_1]{} & A & \xrightarrow[!]{\quad} & * \end{array}$$

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A similar moduli space detects whether a map $f : A \rightarrow B$ is an equivalence

$$\text{isEquiv}(f) := (\Sigma_{g:B \rightarrow A} g \circ f \sim \text{id}_A) \times (\Sigma_{h:B \rightarrow A} f \circ h \sim \text{id}_B)$$

Univalent universes of spaces



Arbitrary equivalences between arbitrary types are elements of a family of spaces

$A \simeq B := \sum_{f:A \rightarrow B} \text{isEquiv}(f)$ indexed by $A, B : \mathcal{U}$ in the universe of (small) spaces.

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Univalence (Voevodsky): Paths in the universe are equivalent to equivalences:

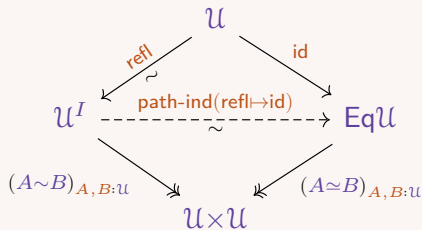
$$\begin{array}{ccc} \mathcal{U}^I & \overset{\sim}{\dashrightarrow} & \text{Eq}\mathcal{U} \\ \downarrow (A \sim B)_{A,B:\mathcal{U}} & & \downarrow (A \simeq B)_{A,B:\mathcal{U}} \\ & \mathcal{U} \times \mathcal{U} & \end{array}$$

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By the principle of **path induction**, to define a map out of the total path space $\mathcal{U}^I$ into a family of spaces over $\mathcal{U} \times \mathcal{U}$, it suffices to define its image on the subspace $\text{refl} : \mathcal{U} \xrightarrow{\sim} \mathcal{U}^I$ of constant paths.

Application of univalence: (higher) group theory



The **structure-identity principle**, a consequence of the univalence axiom, characterizes the path spaces of other moduli spaces constructed from the universe.

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- $\mathbf{Fin}_n$ is a 1-type, with all homotopy groups vanishing above dimension 1.
- Paths $p : X \sim Y$ in $\mathbf{Fin}_n$ are equivalent to equivalences $e : X \simeq Y$.
- The based loop space $\Omega(\mathbf{Fin}_n, \underline{n}) := \underline{n} \sim \underline{n}$ is equivalent to $\Sigma_n := \underline{n} \simeq \underline{n}$.

Consequently $B\Sigma_n := \mathbf{Fin}_n := \Sigma_{X:\mathcal{U}} \|\underline{n} \sim X\|$.

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Consequently $B\Sigma_n := \mathbf{Fin}_n := \sum_{X:\mathcal{U}} \|\underline{n} \sim X\|$.

Higher groups are defined analogously for arbitrary $a : A$ by $B\mathbf{Aut}(a) := \sum_{x:A} \|a \sim x\|$.

An **action** of a higher group BG is a map $X : BG \rightarrow \mathcal{U}$, defining a family $(X_z)_{z:BG}$, with invariants and coinvariants defined by $X^{hG} := \prod_{z:BG} X_z$ and $X_{hG} := \sum_{z:BG} X_z$.



2

Synthetic perspectives on categories (via simplicial type theory)

Simplices from a directed interval



Synthetic category theory is developed in the context of a **directed interval** $\mathbb{2}$:

- with distinct endpoints $0, 1 : \mathbb{2}$,
- with a family of propositions $(x \leq y)_{x,y:\mathbb{2}}$, and
- satisfying reflexivity, transitivity, antisymmetry, totality, with min 0 and max 1 .

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From the interval, (cf. Joyal) we define the standard simplices (and maps between them):

$$\Delta^0 := *$$

$$\Delta^2 := \sum_{x,y:\mathbb{2}} y \leq x$$

$$\Delta^2$$

$$\downarrow (y \leq x)_{x,y:\mathbb{2}}$$

$$\mathbb{2} \times \mathbb{2}$$

$$\Delta^3$$

$$\downarrow (z \leq y \leq x)_{x,y,z:\mathbb{2}} \dots$$

$$\mathbb{2} \times \mathbb{2} \times \mathbb{2}$$

$$\Delta^1 := \mathbb{2}$$

$$\Delta^3 := \sum_{x,y,z:\mathbb{2}} z \leq y \leq x$$

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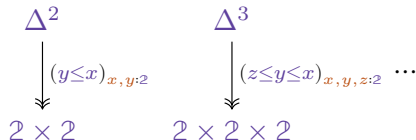
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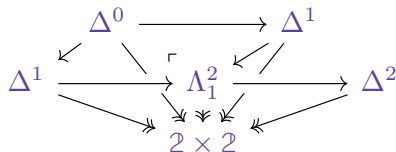
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as well as simplicial subspaces such as

$$\Lambda_1^2 := \sum_{x,y:\mathbb{2}} (y \sim 0) \vee (x \sim 1)$$



Simplicial spaces and precategories



Write “**simplicial spaces**” := synthetic spaces in the context of 2.

Simplicial spaces and precategories



Write “**simplicial spaces**” := synthetic spaces in the context of $\mathcal{U}$.

A simplicial space C has families of

paths

$$\begin{array}{ccc} x \sim y & \longrightarrow & C^I \\ \downarrow & \lrcorner & \downarrow (x \sim y)_{x,y:C} \\ * & \xrightarrow{(x,y)} & C \times C \end{array}$$

and arrows

$$\begin{array}{ccc} \text{Hom}(x, y) & \longrightarrow & C^{\Delta^1} \\ \downarrow & \lrcorner & \downarrow (\text{Hom}(x, y))_{x,y:C} \\ * & \xrightarrow{(x,y)} & C \times C \end{array}$$

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Definition (Riehl–Shulman after Segal): A **precategory** is a simplicial space C in which every composable pair of arrows has a unique composite:

the fibers of

$$\begin{array}{ccc}
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 C^{\Delta^1} & &
 \end{array}$$

over

$$\begin{array}{ccccc}
 & & y & & \\
 & f \nearrow & & \searrow g & \\
 x & & & & z
 \end{array}
 : C^{\Delta^1_1} \text{ are contractible.}$$

Composition, identities, and associativity



Definition (Riehl–Shulman after Segal): A **precategory** is a simplicial space C in which every composable pair of arrows has a unique composite:

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$$\begin{array}{ccc} C^{\Lambda_1^2} & \xrightarrow[\circ]{\quad \quad \quad} & C^{\Delta^1} \\ \searrow (\Sigma_{y:C} \text{Hom}(y,z) \times \text{Hom}(x,y))_{x,z:C} & & \swarrow (\text{Hom}(x,z))_{x,z:C} \\ & C \times C & \end{array}$$

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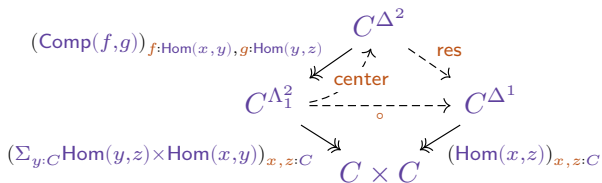
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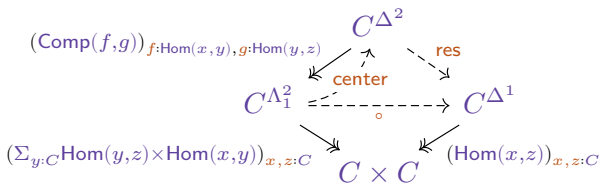
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that is unital and associative, with identities $\text{id}_x : \text{Hom}(x, x)$ defined as constant maps.

Categories and groupoids



An arrow $f : \text{Hom}(x, y)$ in a precategory is an **isomorphism** if

$$\text{isIso}(f) := \left(\sum_{g: \text{Hom}(y, x)} g \circ f \sim \text{id}_x \right) \times \left(\sum_{h: \text{Hom}(y, x)} f \circ h \sim \text{id}_y \right)$$

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Thus, in addition to the families of paths and arrows, precategories have a family of

$$\begin{array}{c} \text{isomorphisms} \quad C^{\mathbb{I}} \\ \downarrow (x \cong y)_{x, y: C} \\ C \times C \end{array}$$

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A precategory C is a **category** if paths are equivalent to isomorphisms:

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Similarly, a simplicial space is a **groupoid** if paths are equivalent to arrows.

Free theorems in synthetic category theory



Definition (Riehl–Shulman after Segal/Rezk): A **category** is a simplicial space C where

- every composable pair of arrows has a unique composite ($C^{\Delta^2} \simeq C^{\Lambda_1^2}$) and
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Free Theorems:

- Functoriality: any map $f : A \rightarrow B$ between categories is a functor.

- Naturality: any map
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \swarrow \\ (A_\gamma)_{\gamma:\Gamma} & \Downarrow & (B_\gamma)_{\gamma:\Gamma} \\ & \Gamma & \end{array}$$
 between **covariant** or **contravariant** families of synthetic categories is a natural transformation.

Path induction and arrow induction



Path Induction (generalizing the indiscernibility of identicals):

For $a : A$, to define a section of a family over the based path space $\Sigma_{x:A} a \sim x$, it suffices to define the image of the constant path refl_a .

$$\begin{array}{ccc} & & P \\ & & \downarrow (P_p)_{p:a \sim x} \\ * & \xrightarrow{\text{refl}_a} & \Sigma_{x:A} a \sim x \end{array}$$

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For $c : C$ a **category**, to define a section of a **covariant** family over the slice category $\Sigma_{x:C} \text{Hom}(c, x)$, it suffices to define the image of the identity arrow id_c .

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A commutative diagram illustrating the Path Induction principle. It features three nodes: a base point $*$ on the left, a type P at the top right, and a path space $\Sigma_{x:A} a \sim x$ at the bottom right. A curved arrow labeled d points from $*$ to P . A straight arrow labeled refl_a points from $*$ to $\Sigma_{x:A} a \sim x$. A vertical arrow points from P down to $\Sigma_{x:A} a \sim x$, labeled with $(P_p)_{p:a \sim x}$ and a dashed line. The text $\text{path-ind}(\text{refl} \mapsto d)$ is written next to the vertical arrow.

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Directed univalence

As **univalence** characterizes the **paths** in the universe of spaces,
 directed univalence characterizes **arrows** in the universes of groupoids or categories.

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Directed univalence (Weaver–Licata, Cisinski–Nguyen, Gratzer–Weinberger–Buchholtz, Cavallo–Riehl–Sattler): Arrows in the universe of groupoids are equivalent to functions:

$$\begin{array}{ccc} & \mathcal{S} & \\ \text{id} \swarrow & & \searrow \text{id} \\ \mathcal{S}^{\Delta^1} & \xrightarrow[\sim]{\text{arr-ind}(\text{id} \mapsto \text{id})} & \text{Fun} \mathcal{S} \\ \searrow & & \swarrow \\ (\text{Hom}(A, B))_{A, B: \mathcal{S}} & \xrightarrow{\quad} & (A \rightarrow B)_{A, B: \mathcal{S}} \\ & \mathcal{S} \times \mathcal{S} & \end{array}$$

Application of directed univalence

Using the directed univalence axiom, we can construct other categories of higher structures and characterize their hom-spaces, via the **structure-homomorphism principle**.

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Example: The simplicial space of magmas is defined by $\mathbf{Magma} := \Sigma_{A:\mathcal{S}} A \times A \rightarrow A$.
By directed univalence:

- $\mathbf{Magma}$ is a category
- an arrow from $(A, \alpha : A \times A \rightarrow A)$ to $(B, \beta : B \times B \rightarrow B)$ is given by a function $f : A \rightarrow B$ with

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ \alpha \downarrow & & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array}$$

Both results use higher directed univalence, an equivalence between $\mathcal{S}^{\Delta^n}$ and n composable functions.

Strange new universes of synthetic mathematics

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WILL MACHINES CHANGE MATHEMATICS?

MAIA FRASER, ANDREW GRANVILLE, MICHAEL H. HARRIS, COLIN MCLARTY,
EMILY RIEHL, AND AKSHAY VENKATESH
(Guest Editorial Committee)

STRANGE NEW UNIVERSES: PROOF ASSISTANTS AND SYNTHETIC FOUNDATIONS

MICHAEL SHULMAN

ABSTRACT. Existing computer programs called proof assistants can verify the correctness of mathematical proofs but their specialized proof languages present a barrier to entry for many mathematicians. Large language models have the potential to lower this barrier, enabling mathematicians to interact with proof assistants in a more familiar vernacular. Among other advantages, this may allow mathematicians to explore radically new kinds of mathematics using an LLM-powered proof assistant to train their intuitions as well as ensure their arguments are correct. Existing proof assistants have already played this role for fields such as homotopy type theory.

“Out of nothing I have created a strange new universe.”

— János Bolyai, one of the inventors of non-Euclidean geometry

“I believe that future mathematics software will combine the ease of interaction of an LLM with the near-absolute trustworthiness of a proof assistant.

...just as a present-day mathematician can exploit a computer’s calculational ability to explore numerical realms undreamed-of decades ago, these imagined future mathematicians may exploit a computer’s logical ability to explore conceptual realms undreamed-of today.

...What new kinds of mathematics, then, may be waiting for us to explore with the proof assistants of tomorrow?”

— *Michael Shulman*

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Thank you!